

## MODEL THEORY FOR THETA-COMPLETE ULTRAPOWERS

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ABSTRACT. We like to develop model theory for  $T$ , a complete theory in  $\mathbb{L}_{\theta,\theta}(\tau)$  when  $\theta$  is a compact cardinal. By [Sh:300a] we have bare bones stability and it seemed we can go no further. Dealing with ultrapowers (and ultraproducts) we restrict ourselves to “ $D$  a  $\theta$ -complete ultrafilter on  $I$ , probably  $(I, \theta)$ -regular”. The basic theorems work, but can we generalize deeper parts of model theory?

In particular can we generalize stability enough to generalize [Sh:c, Ch.VI]? We prove that at least we can characterize the  $T$ ’s which are minimal under Keisler’s order, i.e. such that  $\{D : D \text{ is a regular ultrafilter on } \lambda \text{ and } M \models T \Rightarrow M^\lambda/D \text{ is } \lambda\text{-saturated}\}$ . Further we succeed to connect our investigation with the logic  $\mathbb{L}_{<\theta}^1$  introduced in [Sh:797]: two models are  $\mathbb{L}_{<\theta}^1$ -equivalent iff for some  $\omega$ -sequence of  $\theta$ -complete ultrafilters, the iterated ultra-powers by it of those two models are isomorphic.

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*Date:* March 5, 2013.

2010 *Mathematics Subject Classification.* Primary: 03C45; Secondary: 03C30, 03C55.

*Key words and phrases.* model theory, infinitary logics, compact cardinals, ultrapowers, ultra limits, stability.

The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 1053/11). The author thanks Alice Leonhardt for the beautiful typing. First typed May 10, 2012. Paper 1019.

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[We find for which  $T$ ,  $M^\lambda/D$  is  $(\lambda^+, < \theta, \mathbb{L}_{\theta,\theta})$ -saturated for every model  $M$  of  $T$  and  $\theta$ -complete  $(\lambda, \theta)$ -regular ultrafilter  $D$  on  $\lambda$ . We also characterize a variant dealing with the local version: considering only types having only formulas  $\varphi(\bar{x}, \bar{a})$  with  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,t}$  fixed.]

§3 On  $\mathbb{L}_{<\theta}^1$ , the logic extrapolating  $\mathbb{L}_{\theta,\aleph_0}$  and  $\mathbb{L}_{\theta,\theta}$ , (label d), pg.22

[We characterize being  $\mathbb{L}_{<\theta}^1$ -equivalence of  $M_1, M_2$  by having isomorphic ultralimit by a sequence of length  $\omega$  of  $\theta$ -complete ultrafilters by the logic from [Sh:797].]

## § 0. INTRODUCTION

§ 0(A). **Background and results.** In Winter 2012, I have tried to explain in a model theory class, a position I held for long: model theory can extensively deal with  $\mathbb{L}_{\lambda^+, \aleph_0}$ -classes and a.e.c. but cannot say non-basic things on  $\mathbb{L}_{\lambda, \kappa}$ -classes,  $\lambda \geq \kappa > \aleph_0$ . The latter is known to have downward LST theorems and various connections to large cardinals and consistency results, and only rudimentary stability theory (see [Sh:300b]). Note that, e.g. if  $\mathbf{V} = \mathbf{L}$  there is  $\psi \in \mathbb{L}_{\aleph_1, \aleph_1}$  such that  $M \models \psi$  iff  $M$  is isomorphic to  $(\mathbf{L}_\alpha, \in)$  for some ordinal  $\alpha$  such that  $\beta < \alpha \Rightarrow [\mathbf{L}_\beta]^{\leq \aleph_0} \subseteq \mathbf{L}_\alpha$ ; hence if  $\mu > \text{cf}(\mu) = \aleph_0$  then every  $M$  model of  $\psi$  of cardinality  $\mu$  is isomorphic to  $(\mathbf{L}_\mu, \in)$ .

This work is dedicated to starting to disprove this for the logic  $\mathbb{L}_{\theta, \theta}$ ,  $\theta > \aleph_0$  a compact cardinal.

There was much research in related questions. Recall Kochen use iteration on taking ultra-powers (on a well ordered index set) to characterize elementary equivalence. Gaifman [Gai74] use iteration of ultra-powers on linear ordered index set. Keisler [Kei63] use general  $(\aleph_0, \aleph_0)$ -u.f.l.p., see below, Definition 0.17(1) for  $\kappa = \aleph_0$ . Hodges-Shelah [HoSh:109] is closer to the present work, it deals with isomorphic ultrapowers (and isomorphic reduced powers) for the  $\theta$ -complete case. In particular assuming a  $\theta > \aleph_0$  is a compact cardinal consistently two models have isomorphic ultrapowers for a  $\theta$ -complete ultrafilter iff in all relevant games the anti-isomorphic player does not lose, those were games of length  $\zeta < \kappa$ . Extenders in set theory have been important.

In §1 we start investigating the generalization of Keisler's order, (see [MiSh:996] on background) dealing with saturation of ultra-power for  $\theta$ -complete  $(\lambda, \theta)$ -regular ultrafilters  $D$ . Toward this we have to define  $\lambda^+$ -saturated and then prove the basic property: all models of a (complete)  $T \subseteq \mathbb{L}_{\theta, \theta}(\tau_T)$  behave in the same way; but we shall deal mainly with local saturation.

The main achievement is in §2: a characterization of the minimal theories as stable with  $\theta$ -n.c.p. under reasonable definition. But unlike the first order case, for  $\triangleleft_{\lambda, \theta}$ -maximal  $T$ , we get local saturation for no  $D \in \text{uf}_\theta(\lambda)$ , and some stable theories (even just theories of one equivalence relation) are maximal. In fact we get two characterizations: one for the local version (dealing with types containing  $\varphi(\bar{x}_\varepsilon, \bar{a})$  only for one  $\varphi$ , various  $\bar{a}$ 's) and another for the global one (naturally for theories  $T$ ,  $|T| = \theta$ ).

In §3 we characterize  $\mathbb{L}_{<\theta}^1$ -equivalent with isomorphic iterated ultra-powers by being  $\mathbb{L}'_\theta$ -equivalence, where  $\mathbb{L}'_\theta$  is the logic introduced in [Sh:797].

We may hope this will help us to resolve the categoricity spectrum. It is natural to try to first prove: unstable implies many models. But this is not so - see 1.12; so the situation has a marked difference than the first order case.

This work was presented in a lecture in MAMLS meeting, Fall 2012 and in a course in The Hebrew University, Spring 2012.

*Question 0.1.* 1) For  $\theta > \aleph_0$  a compact cardinal what about  $\mathbb{L}_{\theta, \aleph_0}$  theories?  
2) Consider the logic [HoSh:271, §2], that is, given two compact cardinal  $\kappa > \theta > \aleph_0$ , a logic  $\mathbb{L}_{\lambda/\theta, \lambda/\theta}$  is defined and prove to be “nice”.

On the classical results on  $\mathbb{L}_{\lambda, \kappa}$  see e.g. [Dic85]; on “when for given  $M_1, M_2$  there are  $I$  and  $D \in \text{uf}_\theta(I)$  such that  $M_1^I/D \cong M_2^I/D$ ”, see Hodges-Shelah [HoSh:109].

Recently close works are Malliaris-Shelah [MiSh:999] which deals with  $\kappa$ -complete ultrafilters (on sets and relevant Boolean algebras) on the way to understanding the amount of saturation of ultra-powers by regular ultra-filters.

Concerning dependent (non-elementary) classes, see also [Sh:F1227].

Is the lack of uniqueness of saturation a sign this is a bad choice? It does not seem so to me. If we insist on “union on  $\prec_{\mathcal{L}}$ -increasing countable chain” is an  $\prec_{\mathcal{L}}$ -extension, we can restrict ourselves to  $\mathbb{L}_{\theta}^1$ , but what about unions of length  $\kappa \in \text{Reg} \cap (\aleph_0, \theta)$ ? If we restrict our logice as in  $\mathbb{L}_{\theta}^1$  for all those  $\kappa$  we may get close to a.e.c., or get an interesting new logic with EM models (as indicated in [Sh:797], [Sh:893]). But our intention here is to show  $\mathbb{L}_{\theta, \theta}$  has a model theory, in particular classification theory. At this point having found significant dissimilarities to the first order case on the one hand, and solving the parallel of a serious theorem on the other hand, there is no reason to abandon this direction.

*Question 0.2.* Characterize  $T$  such that  $M^\lambda/D$  is not  $\lambda^+$ -saturated whenever  $\lambda \geq \theta$ ,  $D$  a regular ultrafilter on  $\lambda$ .

*Question 0.3.* 1) Let  $\mathbb{L}_{\kappa}^*$  be  $\{\psi$ : for every  $\mu < \kappa$  large enough we have  $\psi \in \mathbb{L}_{\mu^+, \mu^+}$  and if  $\langle M_s : s \in I \rangle$  is  $\prec_{\mathbb{L}_{\mu^+, \mu^+}}$ -increasing,  $I$  a directed partial order then  $\bigcup_s M_s \models \psi$  iff  $\bigwedge_s M_s \models \psi\}$ . How close is  $\mathbb{L}_{\kappa}^*$  to a.e.c. when  $\kappa$  is a compact cardinal?  
2) As above but  $I$  is linearly ordered.

## § 0(B). Preliminaries.

**Hypothesis 0.4.**  $\theta$  is a compact uncountable cardinal (of course, we use only restricted versions of this).

*Notation 0.5.* 1) Let  $M_1 \prec_{\theta} M_2$  means  $M_1 \prec_{\mathbb{L}_{\theta, \theta}} M_2$  and  $\tau(M_1) = \tau(M_2)$ .  
2)  $\varphi(\bar{x})$  means  $\varphi$  is a formula of  $\mathbb{L}_{\theta, \theta}$ ;  $\bar{x}$  is a sequence of variables with no repetitions including the variables occuring freely in  $\varphi$  and  $\ell g(\bar{x}) < \theta$  if not said otherwise.  
2A) Let  $\bar{x}_{\zeta} = \langle x_{\varepsilon} : \varepsilon < \zeta \rangle$ , etc.  
3)  $T$  denotes a complete theory in  $\mathbb{L}_{\theta, \theta}$ , in the vocabulary  $\tau_T$  with a model of cardinality  $\geq \theta$  if not said otherwise.

*Notation 0.6.* 1)  $\varepsilon, \zeta, \xi$  are ordinals  $< \theta$ .

2) For a linear order  $I$  let  $\text{comp}(I)$  be its completion.

**Definition 0.7.** 1) Let  $\text{uf}_{\theta}(I)$  be the set of  $\theta$ -complete ultrafilters on  $I$ , non-principal if not said otherwise.

2)  $D \in \text{uf}_{\theta}(I)$  is  $(\lambda, \theta)$ -regular when there is a witness  $\bar{w} = \langle w_t : t \in I \rangle$  which means  $w_t \in [\lambda]^{<\theta}$  for  $t \in I$  and  $\alpha < \lambda \Rightarrow \{t : \alpha \in w_t\} \in D$ .

2A) Let  $\text{uf}_{\lambda, \theta}(I)$  be the set of  $(\lambda, \theta)$ -regular  $\theta$ -complete ultrafilters  $D$  on  $I$ .

3) For  $S \subseteq \text{Card} \cap \theta$  and  $D \in \text{uf}_{\theta}(I)$  let  $\text{lcr}_{\theta}(S, D) = \min\{\mu$ : for some  $f \in {}^I S$  we have  $\mu = |\prod_{s \in I} f(s)| \geq \theta\}$ .

4)  $\text{ruf}_{\lambda, \theta}(I)$  is the set  $(\lambda, \theta)$ -regular  $D \in \text{uf}_{\theta}(I)$ , when  $\lambda = |I|$  we may omit  $\lambda$ .

Note that

**Observation 0.8.** If  $S = \text{Card} \cap \theta$  and  $D \in \text{uf}_{\theta}(I)$  and  $\mu$  is the cardinal  $\theta^I/D$  then  $\text{lcr}_{\theta}(S, D)$  is  $\text{Card} \cap \mu^+$  or  $\text{Card} \cap \mu$ .

**Convention 0.9.** A vocabulary  $\tau$  means with  $\text{arity}(\tau) \leq \theta$  where  $\text{arity}(\tau) = \aleph_0 + \sup\{\text{arity}(P) : P \text{ is a predicate (or function symbol) from } \tau\}$ .

**Definition 0.10.** For a set  $v$  of ordinals, a sequence  $\bar{u} = \langle u_\alpha : \alpha \in v \rangle$  and models  $M_1, M_2$  of the same vocabulary  $\tau$  and  $\Delta \subseteq \mathbb{L}_{\theta, \theta}(\tau)$  a set of formulas we define a game  $\mathfrak{D} = \mathfrak{D}_{\Delta, \bar{u}}(M_1, M_2)$  but when  $(\forall \alpha \in v)(u_\alpha = u)$  we may write  $\mathfrak{D}_{\Delta, u, v}(M_1, M_2)$ :

- (a) a play lasts some finite number of moves
- (b) in the  $n$ -th move the antagonist chooses
  - $\alpha_n \in v$  such that  $m < n \Rightarrow \alpha_n < \alpha_m$
  - sequence  $\langle a_{n,i,\ell(n,i)} : i \in u_{\alpha_n} \rangle$  with  $\ell_{n,i} = \ell(n,i) \in \{1, 2\}$  such that
  - $a_{n,i,\ell(n,i)} \in M_{\ell_{n,i}}$
- (c) in the  $n$ -th move (after the antagonist's move) the protagonist chooses  $a_{n,i,3-\ell(n,i)} \in M_{3-\ell(n,i)}$  for  $i \in u_n$
- (d) the play ends when the antagonist cannot choose  $\alpha_n$
- (e) the protagonist wins a play when:
  - the set  $\{(a_{n,i,1}, a_{m,i,2}) : i \in u_{\alpha_m} \text{ and the } m\text{-th move was done}\}$  is a function and even
  - one-to-one function, so a partial one-to-one function from  $M_1$  into  $M_2$  and
  - it preserves satisfaction of  $\Delta$ -formulas and their negations.

We know (see, e.g. [Dic85])

**Claim 0.11.** The  $\tau$ -models  $M_1, M_2$  are  $\mathbb{L}_{\theta, \theta}$ -equivalent iff for every set  $\Delta$  of  $< \theta$  atomic formulas and  $\alpha, \beta < \theta$  the protagonist wins in the game  $\mathfrak{D}_{\Delta, \alpha, \beta}(M_1, M_2)$ .

**Fact 0.12.** For a complete  $T$ .

- 1)  $(\text{Mod}_T, \prec_\theta)$  has amalgamation.
- 2) Types are well defined, see [Sh:300b], i.e. the orbital type **tp** and the types as a set of formula  $\text{tp}_{\mathbb{L}_{\theta, \theta}}$  are essentially equivalent.

The generalization of Los theorem is:

**Theorem 0.13.** If  $\varphi(\bar{x}_\zeta) \in \mathbb{L}_{\theta, \theta}(\tau)$ ,  $D \in \text{uf}_\theta(I)$  and  $M_s$  is a  $\tau$ -model for  $s \in I$  and  $f_\varepsilon \in \prod_{s \in I} M_s/D$  for  $\varepsilon < \zeta$  then  $M \models \varphi[\dots, f_\varepsilon/D, \dots]_{\varepsilon < \zeta}$  iff the set  $\{s \in I : M_s \models \varphi[\dots, f_\varepsilon(s), \dots]_{\varepsilon < \zeta}\}$  belongs to  $D$ .

**Fact 0.14.** Assume  $D \in \text{uf}_\theta(I)$  is not  $\theta^+$ -complete and  $\mathfrak{B} = (\mathcal{H}(\chi), \in, \theta)^I/D$ .

- 1) If  $a_\alpha \in \mathfrak{B}$  for  $\alpha < \theta$  then there is  $\bar{b} \in \mathfrak{B}$  such that  $\bar{b} \models$  “ $\bar{b}$  is a sequence of length  $< \theta$  with the  $\alpha$ -th element being  $a_\alpha$ ” for every  $\alpha < \theta$ .
- 2) If  $D$  is  $(\lambda, \theta)$ -regular and  $a_\alpha \in \mathfrak{B}$  for  $\alpha < \lambda$  then there is  $w \in \mathfrak{B}$  such that  $\alpha < \lambda \Rightarrow \bar{b} \models “|w| < \theta$  and  $a_\alpha \in w”$ .

*Proof.* 1) Let  $a_\alpha = f_\alpha/D$  where  $f_\alpha \in {}^I \mathcal{H}(\chi)$ . Let  $F : I \rightarrow \theta$  be such that  $\alpha < \theta \Rightarrow \{s : \alpha \leq F(s)\} \in D$ , exist by the assumption on  $D$ . We define  $g : I \rightarrow \mathcal{H}(\chi)$  by:

$$\bullet g(s) = \langle f_\alpha(s) : \alpha < F(s) \rangle.$$

Now check.

- 2) similarly using  $\bar{w} = \langle w_s : s \in I \rangle$  from 0.7, so

- $g(s) = \{f_\alpha(s) : \alpha \in w_s\}$ .

□<sub>0.14</sub>

Recall (see more [Sh:863, §5], history [Sh:950, §1])

**Definition 0.15.** Assume  $\Delta \subseteq \mathbb{L}_{\theta, \theta}(\tau_M)$  and  $I$  is a linear order and  $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$  and  $t \in I \Rightarrow \bar{a}_t \in {}^u M$ .

1) We say  $\bar{\mathbf{a}}$  is a middle  $\Delta$ -convergent or strongly  $\Delta$ -convergent in  $M$  when for every  $\varphi(\bar{x}_u, \bar{y}) \in \Delta$  and  $\bar{b} \in {}^{\ell g(\bar{y})} M$  there is  $J \subseteq \text{comp}(I)$  of cardinality  $< \theta$  or  $< \theta_{\varphi(\bar{x}_u, \bar{y})} < \theta$  respectively such that:

- if  $s, t \in I$  and  $\text{tp}_{\text{qf}}(s, J, \text{comp}(I)) = \text{tp}_{\text{qf}}(t, J, \text{comp}(I))$  then  $M \models \varphi[\bar{a}_s, \bar{b}] \equiv \varphi[\bar{a}_t, \bar{b}]$ .

2) We say “strictly  $\Delta$ -convergent” when we can demand “ $J \subseteq I$ ”.

**Definition 0.16.** For a linear order  $I$ .

- 1)  $I^*$  is its inverse.
- 2) A cut is a pair  $(C_1, C_2)$  such that  $C_1$  is an initial segment of  $I$  and  $C_2 = I \setminus C_1$ .
- 3) The cofinality  $(\kappa_1, \kappa_2)$  of the cut  $(C_1, C_2)$  is the pair  $(\kappa_1, \kappa_2)$  of regular cardinals (or 0 or 1) such that  $\kappa_1 = \text{cf}(I \restriction C_1)$ ,  $\kappa_2 = \text{cf}(I^* \restriction C_2)$ .
- 4) We say  $(C_1, C_2)$  is a pre-cut of  $I$  [of cofinality  $(\kappa_1, \kappa_2)$ ] when so is  $(\{s \in I : (\exists t \in C_1)(s \leq_I t), \{s \in I : (\exists t \in C_2)(t \leq_I s)\})$ .

**Definition 0.17.** 0) We say  $X$  respects  $E$  when for some set  $I$ ,  $E$  is an equivalence relation on  $I$  and  $X \subseteq I$  and  $sEt \Rightarrow (s \in X \leftrightarrow t \in X)$ .

1) We say  $\mathbf{x} = (I, D, \mathcal{E})$  is a  $(\kappa, \sigma)$ -u.f.l.p. (ultra-limit parameter) when:

- (a)  $D$  is a filter on the set  $I$
- (b)  $\mathcal{E}$  is a family of equivalence relations on  $I$
- (c)  $(\mathcal{E}, \supseteq)$  is  $\sigma$ -directed, i.e. if  $\alpha(*) < \sigma$  and  $E_i \in \mathcal{E}$  for  $i < \alpha(*)$  then there is  $E \in \mathcal{E}$  refining  $E_i$  for every  $i < \alpha(*)$
- (d) if  $E \in \mathcal{E}$  then  $D/E$  is a  $\kappa$ -complete ultrafilter on  $I$  where  $D/E := \{X/E : X \in D \text{ and } X \text{ respects } I\}$ .

1A) Let  $\mathbf{x}$  is a  $(\kappa, \theta)$ -f.l.p. mean that above we weaken (d) to

- (d)' if  $E \in \mathcal{E}$  then  $D/E$  is a  $\kappa$ -complete filter.

2) Omitting “ $(\kappa, \sigma)$ ” means  $(\theta, \aleph_0)$ .

3) Let  $(I_1, D_1, \mathcal{E}_1) \leq_h^1 (I_2, D_2, \mathcal{E}_2)$  mean that:

- (a)  $h$  is a function from  $I_2$  onto  $I_1$
- (b) if  $E \in \mathcal{E}_1$  then  $h^{-1} \circ E \in \mathcal{E}_2$  where  $h^{-1} \circ E = \{(s, t) : s, t \in I_2 \text{ and } h(s)Eh(t)\}$
- (c) if  $X \in D_2$  then  $h''(X) = \{h(s) : s \in X\}$  belongs to  $D_1$ .

*Remark 0.18.* 1) Note that in 0.17(3), if  $h = \text{id}_{I_2}$  then  $I_1 = I_2$ .

**Definition 0.19.** Assume  $\mathbf{x} = (I, D, \mathcal{E})$  is a  $(\kappa, \sigma)$ -f.l.p.

- 1) For a function  $f$  let  $\text{eq}(f) = \{(s_1, s_2) : f(s_1) = f(s_2)\}$ .
- 2) For a set  $U$  let  $U^I|E = \{f \in {}^I M : \text{eq}(f) \text{ is refined by some } E \in \mathcal{E}\}$ .

- 3) For a model  $M$  let  $\text{flp}_{\mathbf{x}}(M) = M_D^I|_{\mathcal{E}} = (M^I/D) \upharpoonright \{f/D : f \in {}^I M \text{ and } \text{eq}(f) \text{ is refined by some } E \in \mathcal{E}\}$ , pedantically (as  $\text{arity}(\tau_M)$  may be  $> \aleph_0$ ),  $M_D^I/E = \cup\{M_D^I|E : E \in \mathcal{E}\}$ .
- 4) If  $\mathbf{x}$  is u.f.l.p. we may in (3) write  $\text{u.f.l.p.}_{\mathbf{x}}(M)$ .

We now give a generalization of Keisler [Kei63]

**Theorem 0.20.** 1) If  $(I, D, \mathcal{E})$  is  $(\kappa, \sigma)$ -u.f.l.p.,  $\psi \in \mathbb{L}_{\kappa, \sigma}(\tau)$ ,  $\varphi = \varphi(\bar{x}_\zeta) \in \mathbb{L}_{\theta, \sigma}(\tau)$  so  $\zeta < \sigma$ ,  $f_\varepsilon \in M^I|_{\mathcal{E}}$  for  $\varepsilon < \zeta$  then  $M_D^I|_{\mathcal{E}} \models \varphi[\dots, f_\varepsilon/D, \dots]$  iff  $\{s \in I : M \models \varphi[\dots, f_\varepsilon(s), \dots]_{\varepsilon < \zeta}\} \in D$ .

2) Moreover  $M \prec_{\mathbb{L}_{\kappa, \sigma}} M_D^I|_{\mathcal{E}}$ , pedantically  $\mathbf{j} = \mathbf{j}_{M, \mathbf{x}}$  is a  $\prec_{\mathbb{L}_{\kappa, \sigma}}$ -elementary embedding of  $M$  into  $M_D^I|_{\mathcal{E}}$  where  $\mathbf{j}(a) = \langle a : s \in I \rangle/D$ .

3) We define  $(\prod_{s \in I} M_s)_D^I|_{\mathcal{E}}$  similarly when  $\text{eq}(\langle M_s : s \in I \rangle)$  is refined by some  $E \in \mathcal{E}$ , use more in end of the proof of 3.1.

**Convention 0.21.** 1) Abusing a notation in  $\prod_{s \in I} M_s/D$  we allow  $f/D$  for  $f \in \prod_{s \in S} M_s$  when  $S \in D$ .

2) For  $\bar{c} \in \gamma(\prod_{s \in I} M_s/D)$  we can find  $\langle \bar{c}_s : s \in I \rangle$  such that  $\bar{c} \in \ell_{g(\bar{c})}(M_s)$  and  $\bar{c} = \langle \bar{c}_s : s \in I \rangle/D$  which means: if  $i < \ell_{g(\bar{c})}$  then  $c_{s,i} \in M_s$  and  $c_i = \langle c_{s,i} : s \in I \rangle/D$ .

*Remark 0.22.* 1) Why the “pedantically” in 0.20(2)? Otherwise if  $\mathbf{x}$  is a  $(\theta, \sigma)$ -u.f.l.p.,  $(\mathcal{E}_{\mathbf{x}}, \supseteq)$  is not  $\kappa^+$ -directed,  $\kappa < \text{arity}(\tau)$  then defining  $\text{u.f.l.p.}_{\mathbf{x}}(M)$ , we have freedom; if  $R \in \tau$ ,  $\text{arity}_\tau(P) \geq \kappa$ , i.e. on  $R^N \upharpoonright \{\bar{a} : \bar{a} \in {}^{\text{arity}(P)} N \text{ and no } E \in \mathcal{E} \text{ refines } \text{eq}(p)\}$  so we have no restrictions.

2) So for categoricity we better restrict ourselves to vocabularies  $\tau$  such that  $\text{arity}(\tau) = \aleph_0$ .

## § 1. BASIC STABILITY

A difference with the first order case which may be confusing is that the existence of long order is not so strong and does not imply other versions of instability, see Example 1.12.

**Definition 1.1.** 1) We say  $T \subseteq \mathbb{L}_{\theta, \theta}$ , not necessarily complete, is 1-unstable iff for some  $\varepsilon, \zeta < \theta$  and formula  $\varphi(\bar{x}_\varepsilon, \bar{y}_\zeta) \in \mathbb{L}_{\theta, \theta}$  there is a model  $M$  of  $T$  and  $\bar{a}_\alpha \in {}^\varepsilon M, \bar{b}_\alpha \in {}^\zeta M$  for  $\alpha < \theta$  such that  $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta]^{\text{if}(\alpha < \beta)}$  for  $\alpha, \beta < \theta$ .

2) We say  $T \subseteq \mathbb{L}_{\theta, \theta}$  (not necessarily complete) is 4-unstable when there are  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta, \theta}$  and a model  $M$  of  $T$  and  $\bar{b}_\eta \in {}^{\ell g(\bar{y})} M$  for  $\eta \in {}^{\theta > 2}$  such that  $p_\eta(\bar{x}) = \{\varphi(\bar{x}, \bar{b}_{\eta \upharpoonright \alpha})^{\eta(\alpha)} : \alpha < \theta\}$  is a type in  $M$  for every  $\eta \in {}^{\theta > 2}$ .

3) For a class  $\mathbf{I}$  of linear orders we say  $T$  is  $\mathbf{I}$ -unstable when for some  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta, \theta}$  for every  $I \in \mathbf{I}$  there are  $M$  and  $\langle (\bar{a}_s, \bar{b}_s) : s \in I \rangle$  is as in part (1). If  $\mathbf{I} = \{I\}$  we may write  $I$ -unstable.

4) We say  $T$  is strongly/middle  $\mathbf{I}$ -unstable<sup>1</sup> when for some  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta, \theta}$  for every linear order  $I \in \mathbf{I}$  there are  $M \models T$  and strongly/middle convergent sequence  $\langle \bar{a}_s \hat{\ } \bar{b}_s : s \in I \rangle$  such that  $M \models \varphi[\bar{a}_s, \bar{b}_t]^{\text{if}(s < t)}$  for  $s, t \in I$  recalling Definition 0.15(1).

5) We say  $T$  is 3-unstable when it is strongly  $\mathbf{I}_2$ -unstable where  $\mathbf{I}_2 = \{ \sum_{i < i(*)} I_i : i(*) \text{ an ordinal and for each } i, I_i \text{ is anti-isomorphic to some ordinal } \delta_i, \text{cf}(\delta_i) \geq \theta \}.$

6) We say  $T$  is 2-unstable it is  $\mathbf{I}_2$ -unstable.

7) We say  $T$  is 5-unstable if it is  $({}^{\theta > 2}, <_{\text{lex}})$ -unstable.

*Remark 1.2.* We should sort out 1.1(3),(4).

**Definition 1.3.**  $T$  is definably stable (definably unstable is the negation) when : if  $\varphi = \varphi(\bar{x}_\varepsilon, \bar{y}_\zeta) \in \mathbb{L}_{\theta, \theta}$  then there is  $\psi(\bar{y}_\zeta, \bar{z}_\varepsilon) \in \mathbb{L}_{\theta, \theta}$  such that:

- (\*) if  $M \prec_{\mathbb{L}_{\theta, \theta}} N$  are models of  $T$  and  $\bar{a} \in {}^\varepsilon N$  then there is  $\bar{c} \in {}^\varepsilon M$  satisfying:  
 $\psi(\bar{y}_\zeta, \bar{c})$  define  $\text{tp}_\varphi(\bar{a}, M, N)$ , that is:
  - if  $\bar{b} \in {}^\zeta M$  then  $N \models \varphi[\bar{a}, \bar{b}]$  iff  $M \models \psi[\bar{b}, \bar{c}]$ .

**Claim 1.4.** Let  $T \subseteq \mathbb{L}_{\theta, \theta}$ .

1) We have  $(a) \Rightarrow (b) \Rightarrow (x) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Leftrightarrow (h) \Leftrightarrow (i)$  for  $x = c, d$  where:

- (a)  $T$  is 5-unstable, see 1.1(7)
- (b)  $T$  is 4-unstable, see 1.1(2)
- (c) for some  $\varepsilon < \theta$  for every  $\lambda \geq \theta$  there are  $A \subseteq M \models T, |A| = \lambda$  such that  $\mathbf{S}^\varepsilon(A, M) = \{\text{tp}_{\mathbb{L}_{\theta, \theta}}(\bar{a}, A, N), M \prec_\theta N, \bar{a} \in {}^\varepsilon N\}$  has cardinality  $> \lambda$
- (d) for some  $\varepsilon < \theta$ , for every  $\lambda = \lambda^{< \theta} \geq \theta$  there are  $A \subseteq M \models T, |A| = \lambda$  and  $\varphi(\bar{x}_\varepsilon, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_T)$  such that  $\mathbf{S}_{\varphi(\bar{x}_\varepsilon, \bar{y})}^\varepsilon(A, M)$  has cardinality  $> \lambda$
- (e) like (c) but for some  $\lambda = \lambda^{|T|}$
- (f) like (d) but for some  $\lambda = \lambda^{< \theta}$
- (g)  $T$  is definably unstable
- (h) there are  $\varepsilon < \theta, M \models T, \varphi = \varphi(\bar{x}_\varepsilon, \bar{y}_\zeta) \in \mathbb{L}_{\theta, \theta}(\tau_T)$  and  $\langle (\bar{b}_{\alpha, 0}, \bar{b}_{\alpha, 1}) : \alpha < \theta \rangle$  such that:

<sup>1</sup>The difference between 1.1(3) and 1.1(4) is the “convergent”. In part (5) enough when  $\delta_i \in \{\theta, \theta^+\}, i(*) < \lambda$ ; prove a suitable case is enough.



- $\bar{b}_{\alpha,0}, \bar{b}_{\alpha,1} \in {}^\zeta M$  and  $\bar{c}_\alpha \in {}^\varepsilon M$
- $\text{tp}(\bar{b}_{\alpha,0}, \cup\{\bar{b}_\beta, \bar{b}_{\beta,2}, \bar{c}_\beta : \beta < \alpha\}, M) = \text{tp}(\bar{b}_{\alpha,1}, \cup\{\bar{b}_{\beta,1}, \bar{b}_{\beta,2}, \bar{c}_\beta : \beta < \alpha\}, M)$
- $\{\varphi(\bar{x}_\varepsilon, \bar{b}_{\beta,1}), \neg\varphi(\bar{x}_\varepsilon, \bar{b}_{\beta,0}) : \beta < \alpha\}$  is realized by  $\bar{c}_\alpha$

(i)  $T$  is 1-unstable.

2)  $T$  is 5-unstable  $\Rightarrow T$  is 2-unstable  $\Rightarrow T$  is 1-unstable.

3)  $T$  is 1-unstable iff  $T$  is  $\{\lambda\}$ -unstable.

4)  $T$  is 5-unstable iff  $T$  is  $\{I\}$ -unstable for every linearly ordered  $I$ .

5)  $T$  is 2-unstable iff for every  $\varepsilon, \zeta < \theta$  it is  $\varepsilon \times \zeta^*$ -unstable.

*Proof.* 1) (a)  $\Rightarrow$  (b)

Obvious; by clause (a) there is  $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_T)$  which witness  $T$  is  $(\theta^{>2}, <_{\text{lex}})$ -unstable. Let  $\bar{y} = \langle \bar{y}_\zeta, \bar{g}' = \bar{g}_{\zeta+\zeta} \rangle$  and let  $\varphi' = \varphi'(\bar{x}, \bar{y}') = (\varphi(\bar{x}, \bar{y}' \upharpoonright [0, \zeta)) \equiv \varphi(\bar{x}, \bar{y}' \upharpoonright [\zeta, \zeta + \zeta))$ , easily  $\varphi'$  witness  $T$  is 4-unstable.

(b)  $\Rightarrow$  (c)

Let  $\varphi(\bar{x}_\varepsilon, \bar{y}_\zeta)$  be as in 1.1(2), so by compactness for  $\mathbb{L}_{\theta, \theta}$ , for every  $\lambda$  there are  $M_\lambda \models T$  and  $\bar{a}_\nu^\lambda \in {}^\zeta(M_\lambda)$  for  $\nu \in {}^{\lambda>}2$  and  $\bar{c}_\eta^\lambda \in {}^\varepsilon(M_\lambda)$  for  $\eta \in {}^{\lambda>}2$  such that  $M_\lambda \models \varphi[\bar{c}_\eta^\lambda, \bar{a}_\nu^\lambda] \text{ if } (\eta \ell g(\nu))=1$  when  $\nu \triangleleft \eta \in {}^{\lambda>}2$ .

For any cardinal  $\lambda$  let  $\mu = \min\{\mu : 2^\mu > \lambda\}$  hence  $2^{<\mu} \leq \lambda$ , let  $A = \cup\{\bar{a}_\nu^\mu : \nu \in {}^{\mu>}2\} \cup \cup\{\bar{c}_\eta : \eta \in {}^{\mu>}2\}$ , so  $A \subseteq M_\mu$  has cardinality  $\leq 2^{<\mu} + \theta \leq \lambda$  and  $\mathbf{S}^\varepsilon(A, M_\mu)$  has cardinality  $\geq |\{\text{tp}(\bar{c}^\mu, A, M_\mu)_\eta : \eta \in {}^{\mu>}2\}| \geq 2^\mu > \lambda$ .

(c)  $\Rightarrow$  (d)

As  $|\mathbf{S}^\varepsilon(A, M)| \leq \Pi\{|\mathbf{S}_\varphi^\varepsilon(A, M)| : \varphi = \varphi(\bar{x}_\varepsilon, \bar{y}_\zeta) \in \mathbb{L}_{\theta, \theta}(\tau_T)\}$ .

(c)  $\Rightarrow$  (e)

Easy as there are  $\lambda = \lambda^{|T|}$ .

(d)  $\Rightarrow$  (f)

As there are cardinals  $\lambda$  such that  $\lambda = \lambda^{<\theta}$ .

(e)  $\Rightarrow$  (f)

As in (c)  $\Rightarrow$  (d).

(f)  $\Rightarrow$  (g)

By counting.

(g)  $\Rightarrow$  (h)

So by compactness for  $\mathbb{L}_{\theta, \theta}$  for some  $\varepsilon < \theta$  and  $M \models T$  and  $p \in \mathbf{S}^\varepsilon(M)$  and  $\varphi = \varphi(\bar{x}_\varepsilon, \bar{y}_\zeta)$  there is no  $\psi(\bar{y}_\zeta, \bar{z}_\xi)$  as in Definition 1.3. Let  $\tau_* \subseteq \tau_T, |\tau_*| < \theta$  be such that  $\varphi \in \mathbb{L}_{\theta, \theta}(\tau_*)$ .

For each  $\kappa < \theta$  we try by induction on  $\alpha < \kappa$  to choose  $\bar{b}_{\alpha,0}^\kappa, \bar{b}_{\alpha,1}^\kappa, \bar{c}_\alpha^\kappa$  such that:

- $\bar{b}_{\alpha,0}^\kappa, \bar{b}_{\alpha,1}^\kappa \in {}^\zeta M$  realize the same  $\mathbb{L}_{\kappa, \kappa}(\tau_*)$ -type over  $A_\alpha^\kappa := \cup\{\bar{b}_{\beta,0}^\kappa, \bar{b}_{\beta,1}^\kappa, \bar{c}_\beta^\kappa : \beta < \alpha\}$
- $\varphi(\bar{x}_\varepsilon^\kappa, \bar{b}_{\alpha,1}^\kappa), \neg\varphi(\bar{x}_\varepsilon^\kappa, \bar{b}_{\alpha,0}^\kappa) \in p$
- $\bar{c}_\alpha^\kappa$  realizes  $\{\varphi(\bar{x}_\varepsilon^\kappa, \bar{b}_{\beta,1}^\kappa), \neg\varphi(\bar{x}_\varepsilon^\kappa, \bar{b}_{\beta,0}^\kappa) : \beta \leq \alpha\}$ .

**Case 1:** For every  $\kappa$  we succeed to carry the induction.

Let  $\bar{c}^\kappa \in {}^\varepsilon M$  realize  $\{\varphi(\bar{x}_\varepsilon, \bar{b}_{\alpha,1}^\kappa) \wedge \neg(\bar{x}_\varepsilon, \bar{b}_{\alpha,0}^\kappa) : \alpha < \kappa\}$ . By compactness for  $\mathbb{L}_{\theta,\theta}$  we can get clause (g).

**Case 2:** For some  $\kappa$  and  $\alpha < \kappa$ , we cannot choose  $\bar{b}_{\alpha,0}^\kappa, b_{\alpha,1}^\kappa$  (but have chosen  $\langle \bar{b}_{\beta,\ell}^\kappa : \beta < \alpha, \ell < 2 \rangle$ ).

We can find  $\psi$  contradicting our choice of  $M, \varphi, p$ .

(h)  $\Rightarrow$  (i)

Use  $\varphi'$  as in the proof of (a)  $\Rightarrow$  (b) because for  $\alpha, \beta < \theta$  we have  $M \models \varphi[\bar{c}_\alpha \bar{b}_{\beta,a}] \equiv \varphi[\bar{c}_\alpha, \bar{b}_{\beta,2}]$  iff  $\beta > \alpha$ .

(i)  $\Rightarrow$  (g)

Implicit in the proof of 2.13.

2) The arrows are straight.

3) Easy, too.

4) Easy.

5) The  $\Rightarrow$  as  $\theta$  is compact, the  $\Leftarrow$  is trivial.  $\square_{1.4}$

**Conclusion 1.5.** 1) Assume  $T \subseteq \mathbb{L}_{\theta,\aleph_0}$  is (complete and) 3-unstable.

For every  $\lambda = \lambda^{>\theta} > \theta + |T|$ , there are  $M_\alpha \in \text{Mod}_T$  for  $\alpha < 2^\lambda$  which are pairwise non-isomorphic.

*Proof.* By ?? and [Sh:300, Ch.III] or better [Sh:E59, §3].  $\square_{1.5}$

**Question 1.6.** 1) Can we add in 1.5 “pairwise not  $\mathbb{L}_{\theta^+, \theta^+}$ -equivalent”?

2) Phrase  $\mathcal{L}, \mathbb{L}_{\theta,\aleph_0} \subseteq \mathcal{L} \subseteq \mathbb{L}_{\theta,\theta}$  such that  $\psi \in \mathcal{L}(\tau)$  iff  $\psi \in \mathbb{L}_{\theta,\theta}(\tau)$  and for  $\mathbf{t} \in \{\text{yes, no}\}$  the class of models of  $\psi^{\mathbf{t}}$  is closed under  $M_D^I|_{\mathcal{E}}$  when  $(I, D, \mathcal{E})$  is  $(\theta, \aleph_0)$ -complete.

Now recall stability implies the existence of convergence sub-sequences.

**Claim 1.7.** If  $M \equiv_{\mathbb{L}_{\theta,\theta}} N$  then for some  $(\theta, \aleph_0)$ -complete  $(I, D, \mathcal{E})$  we have  $M_D^I/E \cong N_D^I/E$ .

*Proof.* We prove more in 3.3.  $\square_{1.7}$

**Claim 1.8.** Assume  $\lambda = \text{cf}(\lambda)$  and  $\mu < \lambda \Rightarrow \mu^{|T|} + \mu^{<\theta} < \lambda$ . If  $T$  is 1-stable,  $\varepsilon < \theta, M$  is a model of  $T$  and  $\bar{a}_\alpha \in {}^\varepsilon M$  for  $\alpha < \lambda$  then for some stationary  $S \subseteq S_\theta^\lambda$  the sequence  $\langle \bar{a}_\alpha : \alpha \in S \rangle$  is  $(<\omega)$ -indiscernible and  $\mathbb{L}_{\theta,\theta}$ -convergent.

*Proof.* See [Sh:300b].  $\square_{1.8}$

The experience with first order classes say categoricity  $\Rightarrow$  [stability +  $\triangleleft_{\lambda,\theta}$ -minimal] but not so here. We now consider some examples (see also 2.10)

**Example 1.9.** 1) There are  $T = T_1$  and  $T_1^+$  such that:

- (a)  $T \subseteq \mathbb{L}_{\theta,\theta}(\{<\})$  complete
- (b)  $T_1^+ \subseteq \mathbb{L}_{\theta,\theta}(\tau_1^+)$  complete,  $\tau_1^+$  finite
- (c)  $T_1^+ \supseteq T$
- (d) models of  $T$  are dense linear orders
- (e)  $\text{PC}(T, T_1^+)$  is categorical in every  $\lambda \geq \theta$ , recalling
  - $\text{PC}(T, T_1^+) = \{M \upharpoonright \tau_T : M \in \text{Mod}_{T_1^+}\}$

- (f)  $T$  is 2-stable but not 1-stable
- (g)  $T$  is definably stable.

2) Moreover  $T = \text{Th}_{\mathbb{L}_{\theta, \theta}}(N)$  where

- (a)  $N$  is a dense linear order
- (b)  $N$  is the union of  $\aleph_0$  well order sets
- (c)  $N$  has cofinality  $\aleph_0$ , also the inverse
- (d) any two intervals of  $T$  are isomorphic (note:  $T$  cannot say this).

3)  $T_1^+$  above just says in addition only that every two intervals of  $N$  are isomorphic.

*Proof.* See Laver [Lav76] and more in Shelah [Sh:E62], recalling that

(\*)<sub>1</sub> the class of linear orders  $N$  satisfying (a),(b),(c) is closed under submodels

(\*)<sub>2</sub> if  $N$  is a linear order failing (b) of 1.12(2) then there is  $N_1 \subseteq N$  of cardinality  $< \theta$  failing it, hence  $N$  is not a model of  $T$ .

[Why? By  $\theta$  being a compact cardinal.]

□<sub>1.9</sub>

**Conclusion 1.10.**  $T$  being 1-unstable does not imply  $T$  being 2-unstable.

*Proof.* By 1.9.

□<sub>1.10</sub>

**Thesis 1.11.** A big difference with the first order, that is  $\theta = \aleph_0$ , case is:

- (a) long linear orders does not contradict categoricity, see 1.12 below
- (b) interpreting for  $\partial < \theta$ , a group isomorphic to the Abelian group  $(\{\eta \in {}^A 2 : (\exists <^\partial a \in A)(\eta(a) = 1)\}, \Delta)$  appears “for free”
- (c) similarly for the group generated by  $\{x_a : a \in A\}$  freely.

**Example 1.12.** 1) Let  $T_2 = \text{Th}(N)$ ,  $N$  is the linear order  $\theta \times (\theta + 1)^*$  ordered lexicographically.

Then:

- (a)  $T_2$  is 2-unstable but  $T$  is 3-stable as well as 4-stable and 5-stable
- (b)  $M$  is a model of  $T_2$  then  $M$  is  $\sum_{i < \delta} M_i, \delta$  an ordinal of cofinality  $\geq \theta$  and each  $M_i$  is isomorphic,  $\delta_i + 1, \delta_i$  an ordinal of cofinality  $\geq \theta$ .

2) Let  $T_3 = \text{Th}_{\mathbb{L}_{\theta, \theta}}(N)$ ,  $N$  is the linear order  $\theta \times \theta^*$ .

Then

- (a)  $T_3$  is 3-unstable but 4-stable and 5-stable
- (b) like 1.12(1)(b) but  $M_i \cong \delta_i$ .

3) Let  $T_4 = \text{Th}_{\mathbb{L}_{\theta, \theta}}({}^{\theta > 2} 2, \triangleleft)$

- (a)  $T_4$  is 4-unstable but 5-stable, 2-stable and 3-stable
- (b)  $M$  is a model of  $T$  iff it is isomorphic to  $(\mathcal{T}, \triangleleft)$  where for some ordinal  $\alpha$ ,  $\mathcal{T}$  is a subset of  ${}^{\alpha > 2} 2$ , closed under initial segment  $\eta \in \mathcal{T} \Rightarrow \eta \hat{\ } \langle 0 \rangle \in \mathcal{T} \cap \eta \hat{\ } \langle 1 \rangle \in \mathcal{T}$  and  $\mathcal{T}$  is closed under increasing unions of length  $< \theta$ .

4) Let  $T_5 = \text{Th}_{\mathbb{L}_{\theta, \theta}}({}^{\theta > 2} 2, <_{\text{lex}})$

- (a)  $T_5$  is  $\iota$ -unstable, for  $\iota = 1, \dots, 5$ .

## § 2. SATURATION OF ULTRAPOWERS

Note that unlike the first order case, two  $(\lambda, \lambda, \mathbb{L}_{\theta, \theta})$ -saturated models of cardinality  $\lambda$  are not necessarily isomorphic.

*Context 2.1.*  $\theta$  a compact cardinal.

**Definition 2.2.** 1) We say  $M$  is  $(\lambda, \sigma, \mathcal{L})$ -saturated (where  $\mathcal{L}$  is a logic; the default value is  $\mathcal{L} = \mathbb{L}_{\theta, \theta}$ ) when if  $\Gamma$  is a set of  $< \lambda$  formulas from  $\mathcal{L}$  with parameters from  $M$  with  $< 1 + \sigma$  free variables, and  $\Gamma$  is  $(< \theta)$ -satisfiable in  $M$ , then  $\Gamma$  is realized in  $M$ . If  $\sigma = \theta$  we may omit it and  $\leq \sigma$  means  $\sigma^+$ .

2) We say “locally” when using one  $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathcal{L}$ , i.e. all members of  $\Gamma$  have the form  $\varphi(\bar{x}, \bar{b})$ .

3) We say “fully  $(\lambda, \mathcal{L})$ -saturated” when  $\sigma = \lambda$ .

**Claim 2.3.** 1) If  $D \in \text{uf}_\theta(I)$  is  $(\lambda, \theta)$ -regular and  $M_1, M_2$  are  $\mathbb{L}_{\theta, \theta}$ -equivalent and  $\gamma < \lambda^+$  then  $M_1^I/D, M_2^I/D$  are  $\mathbb{L}_{\lambda^+, \lambda^+}$ -equivalent, moreover  $\mathbb{L}_{\infty, \lambda^+, \gamma}$ -equivalent (so one is  $(\lambda^+, \sigma^+, \mathbb{L}_{\theta, \theta})$ -saturated iff the other is).

*Remark 2.4.* Recall that  $\mathbb{L}_{\chi, \mu, \gamma}(\tau) = \{\varphi(\bar{x}) \in \mathbb{L}_{\chi, \mu}(\tau) : \varphi(\bar{x}) \text{ has quantifier depth } < \gamma\}$ .

*Proof.* As  $D$  is  $(\lambda, \theta)$ -regular there is a sequence  $\langle (u_s, v_s, \Delta_s) : s \in I \rangle$  such that  $v_s \in [\gamma]^{< \theta}, u_s \in [\lambda]^{< \theta}, \Delta_s$  a set of  $< \theta$ -formulas of  $\mathbb{L}_{\theta, \theta}(\tau_T)$  and  $\alpha < \gamma \wedge \beta < \lambda \wedge \varphi(\bar{x}) \in \mathbb{L}_{\theta, \theta}(\tau_T) \Rightarrow \{s : \alpha \in v_s, \beta \in u_s \text{ and } \varphi(\bar{x}) \in \Delta_s\} \in D$ .

For  $s \in I$  let  $\mathfrak{D}_s$  be the game  $\mathfrak{D}_{\Delta_s, u_s, v_s}(M_1, M_2)$ , see Definition 0.10. As  $M_1, M_2$  are  $\mathbb{L}_{\theta, \theta}$ -equivalent by 0.11 the protagonist wins this game  $\mathfrak{D}_s$  hence has a winning strategy  $\mathbf{st}_s$ . Let  $N_\ell = M_\ell^I/D$  and it suffices to find a strategy  $\mathbf{st}$  for the protagonist in the game  $\mathfrak{D}_{\mathbb{L}_{\theta, \theta}, \lambda, \gamma}$ . The strategy is obvious, see details of such a proof in 3.3.

□<sub>2.3</sub>

**Definition 2.5.** 1) We define a two-place relation  $\triangleleft_{\lambda, \theta}$  on the class of complete theories  $T$  (in  $\mathbb{L}_{\theta, \theta}$ , of course) of cardinality  $\leq \lambda$ . We have  $T_1 \triangleleft_{\lambda, \theta} T_2$  iff for every  $D \in \text{ruf}_\theta(\lambda)$  and models  $M_1, M_2$  of  $T_1, T_2$ , respectively we have: if  $M_2^\lambda/D$  is locally  $(\lambda^+, \mathbb{L}_{\theta, \theta})$ -saturated then so is  $M_1^\lambda/D$ .

2) We say fully or write  $\triangleleft_{\lambda, \theta}^{\text{ful}}$ , when we deal with full saturation.

**Conclusion 2.6.** In Definition 2.5 the choice of  $M_1, M_2$  does not matter.

*Proof.* By 2.3.

□<sub>2.6</sub>

**Claim 2.7.** 1)  $\text{Th}_{\mathbb{L}_{\theta, \theta}}((\theta, <))$  is a  $\triangleleft_{\lambda, \theta}$ -maximal theory.

2)  $\text{Th}_{\mathbb{L}_{\theta, \theta}}(\theta, =)$  is a  $\triangleleft_{\lambda, \theta}$ -minimal theory.

*Proof.* 1) Easy: we never get saturation.

2) Easy: the (full)  $(\lambda^+, \lambda^+, \mathbb{L}_{\theta, \theta})$ -saturated means just “of cardinality  $\geq \lambda^+$ ”. □<sub>2.7</sub>

**Definition 2.8.** 1) We say  $T$  has the  $\theta$ -n.c.p. when it fails the  $\theta$ -c.p. which means: for some  $\varphi = \varphi(\bar{x}_\varepsilon, \bar{y}_\zeta) \in \mathbb{L}_{\theta, \theta}(\tau_T)$  so  $\varepsilon, \zeta < \theta$  for every  $\partial < \theta$  there are a model  $M$  of  $T$  and  $\Gamma$  such that:

- $\Gamma \subseteq \{\varphi(\bar{x}_\varepsilon, \bar{b}) : \bar{b} \in {}^\zeta M\}$
- $|\Gamma| < \theta$
- $\Gamma$  is  $(< \partial)$ -satisfiable in  $M$

- $\Gamma$  is not satisfiable in  $M$ .

2) Let  $\text{spec}(\varphi, T) = \{\partial < \theta : \text{there is } \Gamma \text{ as above of cardinality } \partial\}$ .

3) For  $\varepsilon < \theta$ , if  $\Delta \subseteq \Phi_{T, \varepsilon} := \{\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_\varphi) : \varphi \in \mathbb{L}_{\theta, \theta}(\tau_T)\}$  we define the  $\text{spec}(\Delta, T)$  as the set of cardinals  $\partial < \theta$  such that for some model  $M$  of  $T$  and sequence  $\langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_{\varphi_\alpha}) : \alpha < \partial \rangle$  of members of  $\Delta$  and  $\bar{a}_\alpha \subseteq M$  of length  $\ell g(y_{\varphi_\alpha})$  for  $\alpha < \partial$ , the set  $\{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha) : \alpha < \partial\}$  is not realized in  $M$  but any subset of smaller cardinality is realized.

4) We may replace  $\Delta$  by a sequence listing its members (even with repetitions).

**Observation 2.9.** 1)  $T$  has  $\theta$ -c.p. iff for some  $\varphi$ ,  $\text{spec}(\varphi, T)$  is unbounded in  $\theta$  iff for some  $\varepsilon < \theta$  and  $\Delta \subseteq \Phi_{T, \varepsilon}$  of cardinality  $< \theta$  the set  $\text{spec}(\Delta, T)$  is unbounded in  $\theta$ .

2) In the definition of “the theory  $T$  has the  $\theta$ -c.p.”, see Definition 2.8, the model  $M$  does not matter.

3) If  $\varepsilon < \theta$  and  $\Delta \subseteq \Phi_{T, \varepsilon}$  has cardinality  $< \theta$  then for some  $\psi = \psi(\bar{x}_{[\varepsilon]}, \bar{y}_\psi)$  we have:

- (a)  $\text{spec}(\Delta, T) \subseteq \text{spec}(\psi, T)$
- (b) if  $M \models T$  then  $\{\emptyset\}, \{\varphi(M, \bar{a}) : \varphi(\bar{x}_{[\varepsilon]}, \bar{y}) \in \Delta \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}M\} = \{\psi(M, \bar{a}) : \bar{a} \in {}^{\ell g(\bar{y})}\psi, M\}$ .

*Proof.* 1) The second assertion implies the first and the third trivially implies the first by part (3) so we are left with proving “the first implies the second”.

For  $\partial < \theta$ , let  $\Gamma$  be as in 2.8(1) for  $\partial$ , so necessarily  $|\Gamma| \geq \partial$ , let  $\Gamma_1 \subseteq \Gamma$  be of minimal cardinality such that  $\Gamma_1$  is not realized in  $M$ . So  $|\partial| \leq |\Gamma_1| \in \text{spec}(\varphi, T)$ .

2) Read Definition 2.8.

3) Use definition by cases, ignoring  $T$  which has a model with just one element.

□<sub>2.9</sub>

For first order  $T$ ,  $\aleph_0$ -c.p. = fcp follows from instability, but not so here.

**Claim 2.10.** *There are 5-unstable  $T$  with  $\theta$ -n.c.p. which are no 3-unstable.*

*Proof.*  $T$  be the theory of  $I$  for any dense linear order which is  $\theta$ -saturated (in the first order sense) with neither first nor last member.

□<sub>2.10</sub>

More generally

**Example 2.11.**  $T = \text{Th}_{\mathbb{L}_{\theta, \theta}}(M)$ ,  $M$  a  $\theta$ -saturated model (in the first order sense) with  $\text{Th}_{\mathbb{L}}(M)$ , the first order theory of  $M$  being unstable (e.g. random graph).

Clearly

- (a)  $T$  is 5-unstable
- (b)  $T$  has  $\theta$ -n.c.p.

**Claim 2.12.** *The model  $N = M^I/D$  is not  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated (even locally, and even just for  $\varphi$ -types) when:*

- (a)  $D \in \text{uf}_\theta(I)$
- (b)  $\varphi(\bar{x}_\varepsilon, \bar{y}_\zeta)$  witnesses  $T$  has the  $\theta$ -c.p.
- (c)  $\chi = \text{lcr}_\theta(\text{spec}(\varphi, T), D)$  see 0.7(3), equivalently letting  $(J, <_J, P^J) = (\theta, <, \text{spec}(\varphi, T))^I/D$  we have  $\chi = \min\{|\{s : s <_J t\}| : t \in P^J, \text{ but } (\exists^{\geq \theta} s)(s <_J t)\}$ .

*Proof.* Straightforward or see the proof of 2.22.  $\square_{2.12}$

**Claim 2.13.** *The model  $N = M^I/D$  is not  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated when:*

- (a)  $D \in \text{ruf}_\theta(I)$
- (b)  $T = \text{Th}(M)$  is 1-unstable
- (c)  $\chi = \text{cf}(\theta^I/D)$ , so  $\chi > \theta$ .

*Remark 2.14.* Recall the frames in the proof of (i)  $\Rightarrow$  (g) in 1.4.

*Proof.* By 2.3 it suffice to find some model  $M$  of  $T$  such that  $N = N_M := M^I/D$  is not  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated. Toward contradiction assume that every such  $N$  is  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated.

Fix  $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta, \theta}$  witnessing  $T$  is not 1-stable hence we can choose

- $\otimes_1$  (a) a model  $M$  of  $T$
- (b)  $\bar{a}_\alpha^1 \in {}^\varepsilon M, \bar{b}_\beta^1 \in {}^\zeta M$  for  $\alpha, \beta < \theta$  are such that  $M \models \varphi[\bar{a}_\alpha^1, \bar{b}_\beta^1]^{\text{if}(\alpha < \beta)}$
- $\otimes_2$  if possible then:
  - (a) for every  $\bar{a} \in {}^\varepsilon M$  for some truth value  $\mathbf{t}$  for every  $\beta < \theta$  large enough we have  $M \models \varphi[\bar{a}, \bar{b}_\beta^1]^{\mathbf{t}}$
  - (b) for every  $\bar{b} \in {}^\zeta M$  for some truth value  $\mathbf{t}$  for every  $\alpha < \theta$  large enough we have  $M \models \varphi[\bar{a}_\alpha^1, \bar{b}]^{\mathbf{t}}$ .

So

- $\otimes_3$   $N = M^I/D$  is well defined and  $M \prec_\theta N$ , i.e. we identify  $a \in M$  with  $\langle a : s \in I \rangle/D$ .

Case 1: In  $\otimes_2$  the answer is impossible and  $|\tau_T| \leq \theta$ .

Choose  $D_* \in \text{uf}_\theta(\theta)$ .

We choose  $(N_\alpha, \bar{a}_\alpha^2, \bar{b}_\alpha^2)$  by induction on  $\alpha < \theta$  such that:

- $(*)_\alpha^1$  (a)  $N_\alpha \prec_{\mathbb{L}_{\theta, \theta}} N$  has cardinality  $\theta$
- (b)  $\beta < \alpha \Rightarrow N_\beta + \bar{a}_\beta^2 + \bar{b}_\beta^2 \subseteq N_\alpha$
- (c)  $\bar{a}_\alpha^2$  realizes  $\{\varphi(\bar{x}_\varepsilon, \bar{b})^{\mathbf{t}} : \bar{b} \in {}^\zeta(N_\alpha) \text{ and } \{\beta < \kappa : \varphi(\bar{a}_\beta^1, \bar{b})^{\mathbf{t}}\} \in D_* \text{ and } \mathbf{t} \in \{0, 1\}\}$
- (d)  $\bar{b}_\alpha^2$  realizes  $\{\varphi(\bar{a}, \bar{y}_\zeta)^{\mathbf{t}} : \bar{a} \in {}^\varepsilon(N_\alpha + \bar{a}_\alpha^2) \text{ and } \{\beta < \kappa : \varphi(\bar{a}, \bar{b}_\beta^1)^{\mathbf{t}}\} \in D_* \text{ and } \mathbf{t} \in \{0, 1\}\}$ .

There is no problem to carry this by our assumption toward contradiction on  $N$ . Let  $M_1 = \cup\{N_\alpha : \alpha < \theta\}$ , so  $M_1 \prec_{\mathbb{L}_{\theta, \theta}} N$ . Now  $\langle (\bar{a}_\alpha^2, \bar{b}_\alpha^2) : \alpha < \theta \rangle$  and  $M_1$  exemplifies that the answer to  $\otimes_2$  is yes, so the present case is done.

Case 2: The answer in  $\otimes_2$  is yes and  $|\tau_T| \leq \theta$ .

Choose  $D_{**} \in \text{uf}_{\lambda, \theta}(\lambda)$ .

Let  $M^+ = (M, P^{M^+}, <^{M^+})$  where  $P^{M^+} = \{\bar{a}_\alpha^1 \wedge \bar{b}_\alpha^1 : \alpha < \theta\}$ ,  $<^{M^+} = \{\bar{a}_\alpha^1 \wedge \bar{b}_\alpha^1 \wedge \bar{a}_\beta^1 \wedge \bar{b}_\beta^1 : \alpha < \beta < \theta\}$  and  $N^+ = (M^+)^I/D$  hence clearly  $N^+ = (M^I/D, P^{N^+}, <^{N^+})$ . Clearly  $(P^{N^+}, <^{N^+})$  is a linear order of cofinality  $\chi$  so we can choose an increasing cofinal sequence  $\langle \bar{a}_\alpha^3 \wedge \bar{b}_\alpha^3 : \alpha < \chi \rangle$ , and by 0.13

(\*)<sub>2</sub> if  $\bar{a} \in {}^\varepsilon |N^+|$  then for some truth value  $\mathbf{t}$  for every  $\alpha < \chi$  large enough  $N^+ \models \varphi[\bar{a}, \bar{b}_\alpha^3]^\mathbf{t}$ ; of course this is a property of  $N$ .

Now choose  $(N_\alpha^3, \bar{a}_\alpha^4, \bar{b}_\alpha^4)$  by induction on  $\alpha < \chi$  as in  $(*)_\alpha^1$  with  $\chi, D_{**}, \langle (\bar{a}_\beta^3 \wedge \bar{b}_\beta^3) : \beta < \chi \rangle, N_\alpha^3, \bar{a}_\alpha^4, \bar{b}_\alpha^4$  here standing for  $\theta, D_*, \langle \bar{a}_\beta^1, \bar{b}_\beta^1 \rangle : \beta < \theta \rangle, N_\alpha, \bar{a}_\alpha^2, \bar{b}_\alpha^2$  there; hence so  $M_3 := \cup \{N_\alpha^3 : \alpha < \chi\}$  is  $\prec_\theta N$ . Now  $(M_3, \langle (\bar{a}_\alpha^3, \bar{b}_\alpha^4) : \alpha < \chi \rangle)$ , recalling that necessarily  $\chi = \chi^{<\theta}$ , satisfies:

- $\boxplus_{M_3, \langle (\bar{a}_\alpha^3, \bar{b}_\alpha^3, \bar{b}_\alpha^4) : \alpha < \chi \rangle}^\chi$  (a)  $M_3$  is a model of  $T$  of cardinality  $\chi$   
 (b)  $\bar{b}_\alpha^3, \bar{b}_\alpha^4 \in {}^\zeta(M_3)$  and  $\bar{a}_\alpha^3 \in {}^\varepsilon(M_3)$   
 (c) if  $\bar{a} \in {}^\varepsilon(M_3)$  then for every  $\alpha < \chi$  large enough for some truth value  $\mathbf{t}$  we have  $M_3 \models \varphi[\bar{a}, \bar{b}_\alpha^3]^\mathbf{t} \wedge \varphi[\bar{a}, \bar{b}_\alpha^4]^\mathbf{t}$   
 (d)  $M_3 \models \varphi[\bar{a}_\alpha^3, \bar{b}_\beta^4]$  for  $\alpha, \beta < \chi$   
 (e) if  $\alpha, \beta < \chi$  then  $M_3 \models \varphi[\bar{a}_\alpha^3, \bar{b}_\beta^3]$  iff  $\alpha < \beta$ .

As  $|\tau_T| \leq \theta$  by the downward LST theorem there are  $M_4 \prec_\theta M_3$  of cardinality  $\theta$  and an increasing sequence  $\langle \alpha(i) : i < \theta \rangle$  of ordinals  $< \chi$  such that  $(M_4, \langle (\bar{b}_\alpha^3, \bar{a}_\alpha^3, \bar{b}_{\alpha(i)}^4) : i < \theta \rangle)$  satisfies the parallel of  $\boxplus_{M_3, \langle (\bar{a}_\alpha^3, \bar{b}_\alpha^3, \bar{b}_\alpha^4) : \alpha < \theta \rangle}^\chi$ .

Now it is easy to see that  $M_4^I/D$  is not locally  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated.

Case 3:  $|\tau_T| > \theta$

We can find  $\tau_* \subseteq \tau_T, |\tau_*| \leq \theta$  such that  $T_* = T \cap \mathbb{L}_{\theta, \theta}(\tau_*)$  (is complete and) satisfies of assumptions from the claim. So by cases 1,2 we know that for some model  $M_*$  of  $T_*$ ,  $M_*^I/D$  is not locally  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated hence by 2.3 this holds for every model of  $T_*$  hence it holds for every model of  $T$ .  $\square_{2.13}$

**Theorem 2.15.** Assume  $T$  is complete of cardinality  $\theta$  and has  $\theta$ -n.c.p. and is definably stable and  $\lambda = \lambda^{<\theta}$ .

- 1)  $T$  is (locally)  $\triangleleft_{\lambda, \theta}$ -minimal.
- 2) Moreover, if  $D \in \text{uf}_{\lambda, \theta}(I)$  and  $\theta^I/D > \lambda$  and  $M \models T$  then  $M^I/D$  is locally  $(\lambda^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated.

*Proof.* 1) By part (2).

2) Without loss of generality  $|\tau_T| \leq \theta$ .

Let  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta, \theta}$  and  $\partial = \partial_\varphi < \theta$  witness  $\varphi(\bar{x}, \bar{y})$  fail the  $\theta$ -c.p. and let  $\varepsilon = \ell g(\bar{x}), \zeta = \ell g(\bar{y})$  and  $N = M^I/D$ , where  $D \in \text{uf}_\theta(\lambda)$  and  $M$  is a model of  $T$  and  $p(\bar{x}) = p_0(\bar{x})$  is a  $\varphi$ -type in  $N$  of cardinality  $\leq \lambda$ , so  $p(\bar{x}) \subseteq \{\varphi(\bar{x}, \bar{b})^\mathbf{t} : b \in {}^{\ell g(\bar{y})} N \text{ and } \mathbf{t} \in \{0, 1\}\}$  is  $(< \theta)$ -satisfiable in  $N$ .

As  $\theta$  is compact there is  $p_1(\bar{x}) \in \mathbf{S}_\varphi^\varepsilon(N)$  extending  $p_0(x)$ . By Definition 2.5 there is  $\psi(\bar{y}, \bar{z}) \in \mathbb{L}_{\theta, \theta}(\tau_T)$  and  $\bar{c} \in {}^{\ell g(\bar{z})} N$  which defines  $p_1(\bar{x})$ . Let  $\bar{c}_s \in {}^{\ell g(\bar{z})} M$  for  $s \in I$  be such that  $\bar{c} = \langle \bar{c}_s : s \in I \rangle / D$  and for  $s \in I$  let  $\Gamma_s = \{\varphi(x, \bar{b})^\mathbf{t} : M \models \psi[\bar{b}, \bar{c}_s]^\mathbf{t} \text{ and } \mathbf{t} \in \{0, 1\}\}$ .

Let  $I_\partial = \{s : \Gamma_s \text{ is } (< \partial)\text{-satisfiable in } M_s, \text{ that is if } \bar{b}_\alpha \in {}^\zeta(M_\alpha), M_s \models \psi[\bar{b}_\alpha, \bar{c}_s]^\mathbf{t} \text{ for } \alpha < \partial \text{ then } M \models \exists \bar{x} \bigwedge_{\alpha < \partial} \varphi(\bar{x}, \bar{b}_\alpha)^\mathbf{t} \text{ for } \mathbf{t} \in \{0, 1\}\}$ ; so by 0.13 necessarily  $I_\partial \in D$ .

By the choice of  $\partial$  and of  $I_\partial$  for every  $s \in I_\partial$  we have  $\Gamma_s \in \mathbf{S}_\varphi^\varepsilon(M_1)$ .

Let  $\chi$  be large enough such that  $M \in \mathcal{H}(\chi)$  and let  $\mathfrak{B} = (\mathcal{H}(\chi), \in, M)^I/D$ . As  $s \in J \Rightarrow \Gamma_s \in \mathcal{H}(\chi)$  we have  $\Gamma = \langle \Gamma_s : s \in I \rangle / D \in \mathfrak{B}$  and  $\mathfrak{B} \models \text{"}\Gamma \text{ is a complete"}$



$\varphi$ -type over  $M$ ". Let  $\Gamma' = \{\varphi(\bar{x}, \bar{a}) : \mathfrak{B} \models \text{"}\varphi(\bar{x}, \bar{a}) \in \Gamma\text{"}\}$ . Hence to prove  $p_0(\bar{x})$  is realized it suffices to show

- there is  $w \in \mathfrak{B}$  such that  $\varphi(\bar{x}, \bar{b}) \in p_0(x) \Rightarrow \mathfrak{B} \models \text{"}\bar{b} \in w \text{ and } |w| < \theta\text{"}$ .

By 0.14(2) this holds.  $\square_{2.15}$

**Conclusion 2.16.** *Assume  $\lambda \geq 2^\theta$  and  $|T| \leq \theta$ , then  $T$  is  $\triangleleft_{\lambda, \theta}$ -minimal iff  $T$  is 1-stable with  $\theta$ -n.c.p.*

*Proof.* Case 1:  $T$  has the  $\theta$ -c.p.

Let  $D_1 \in \text{ruf}_\theta(\lambda)$  and  $D_2$  be a normal ultrafilter on  $\theta$  and so  $D = D_1 \times D_2 \in \text{ruf}_\theta(\lambda \times \theta)$ . If  $M \models T$  then  $M^{\lambda \times \theta}/D \cong (M^\lambda/D_1)^\theta/D_2$ ; let  $M_0 = M, M_1 = M_0^\lambda/D$  and  $M_2 = M_1^\theta/D$ , all models of  $T$ . So  $M^{\lambda \times \theta}/D$  is isomorphic to  $M_1^\theta/D$  and the latter is not locally  $((2^\theta)^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated by 2.12, (hence not  $(\lambda^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated).

Case 2:  $T$  is 1-unstable

Let  $D, D_1, D_2, M_0 = M, M_1, M_2$  be as in Case 1.

Now apply Claim 2.13 to the model  $M_1^\lambda/D_1$ , so it is not locally  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated where  $\chi = \text{cf}(\theta, \ell)^\theta/D_2$ . But  $(\theta, <)^\theta/D_2$  has cofinality  $\leq 2^\theta$  so we are done.

Case 3:  $T$  is 1-stable with  $\theta$ -n.c.p.

Use 2.15 recalling  $T$  is definable stable by 1.4(1), the  $(g) \Rightarrow (i)$ .  $\square_{2.16}$

**Conclusion 2.17.** *If  $T$  is  $\theta$ -n.c.p. then  $T$  is 1-stable iff  $T$  is definably stable.*

**Claim 2.18.** 1) *If  $\text{spec}(\varphi(\bar{x}, \bar{y}), T) = \theta$  and  $\lambda \geq \theta$  then  $T$  is a  $\triangleleft_{\lambda, \theta}$ -maximal.*

2) *There is a model  $M_* = (\theta, E^M), E^M$  an equivalence relation such that  $T = \text{Th}_{\mathbb{L}_{\theta, \theta}}(M)$  satisfies  $\text{spec}(xEy, T) = \theta \cap \text{Card}$  hence  $T$  is  $\triangleleft_{\lambda, \theta}$ -maximal for every  $\lambda$ .*

3) *Assume  $\kappa$  is supercompact with Laver diamond. There is a sequence of models  $\langle M_A : A \subseteq \theta \rangle$  such that:*

- (a)  $M_A = (\theta, E_A)$  for  $A \subseteq \theta, E_A$  an equivalence relation on  $\theta$  such that letting  $T_A = \text{Th}(M_A)$  we have
- (b) for  $\lambda = \lambda^{<\theta}, T_A \triangleleft_{\lambda, \theta} T_B$  iff  $A \subseteq B$ .

*Proof.* 1) By 2.12, because for  $\theta$ -complete which just not  $\theta^+$ -complete<sup>2</sup> ultrafilter on  $I$  we know that  $\theta \in \{\prod_{s \in I} \theta_s/E : \theta_2 < \theta\}$ .

2) E.g.  $E^M = \{(\alpha, \beta) : \alpha + |\alpha| = \beta + |\beta|\}$  satisfies the first demand; the “hence” follows by (1).

3) Let  $C = \{\mu : \mu < \theta \text{ is strong limit}\}$ , let  $\langle S_i : i < \theta \rangle$  be a partition of  $C$  to  $\theta$  unbounded subsets of  $C$  such that for each  $i$  there is a normal ultrafilter  $D_i^*$  on  $\theta$  which  $S_i$  belongs. Fill reference for existence. For  $A \subseteq \theta$ , let  $E_A$  be an equivalence relation on  $\theta$  such that  $\{(\alpha/E_A) : \alpha < \theta\} = \cup\{S_i : i \in A\}$ . So the following claim 2.19 suffice.  $\square_{2.18}$

**Claim 2.19.** *Assume  $\theta < \lambda = \lambda^{<\theta}$  and  $f_* : \theta \rightarrow \theta$  satisfies  $\alpha < \theta \Rightarrow \alpha < f_*(\alpha) \in \text{Card}$  and there is transitive  $\mathbf{M} \supseteq {}^\lambda \mathbf{M}$  and an elementary embedding  $\mathbf{j}$  of  $\mathbf{V}$  into  $\mathbf{M}$  with critical point  $\lambda$  such that  $(\mathbf{j}(f_*))(\theta) = \lambda$ .*

*Let  $E$  be a thin enough club of  $\theta, S_1 = \text{Rang}(f_* \upharpoonright E)$  and let  $S_2 = \{2^\mu : \mu \in S_1\}$ .*

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<sup>2</sup>being  $(\lambda, \theta)$ -regular is a stronger condition

Then there is  $D \in \text{ruf}_\theta(\lambda)$  such that we have:

- (a) if  $f : \lambda \rightarrow S_1$  then the cardinal  $\prod_{\alpha < \lambda} f(\alpha)/D$  is  $< \theta$  or is  $\geq \lambda$
- (b) for some  $f : \lambda \rightarrow S_1$  we have  $\prod_{\alpha < \lambda} f(\alpha)/D$  is  $\lambda$
- (c) if  $f : \lambda \rightarrow S_2$  then the cardinality  $\prod_{\alpha < \lambda} f(\alpha)/D$  is  $< \theta$  or is  $> \lambda$
- (d) for some  $f : \lambda \rightarrow S_2$  we have  $\prod_{\alpha < \lambda} f(\alpha)/D$  is  $2^\lambda$ .

*Proof.* Let  $S_0 = \{\mu < \theta : \mu \text{ strong limit, } f_*(\mu) \text{ is a cardinal } > \mu \text{ and } \text{Rang}(f_* \upharpoonright \mu) \subseteq \mathcal{H}(\mu)\}$ . Let  $S_1 = \{f_*(\mu) : \mu \in S_0\}$  and  $S_2 = \{2^{f_*(\mu)} : \mu \in S_1\}$ .

Let  $D$  be the following normal ultrafilter on  $I = [\lambda]^{<\theta}$

$$\{\mathcal{U} \subseteq {}^\theta \lambda : \{\mathbf{j}(\alpha) : \alpha < \lambda\} \in \mathbf{j}(\mathcal{U})\}$$

hence the following set belongs to  $D$ :  $\{s \in [\lambda]^{<\theta} : s \in S, s \cap u \in I \text{ and } |u| = f_*(s \cap \theta)\}$ .

Clearly  $D$  is a  $\theta$ -complete  $(\lambda, \theta)$ -regular ultrafilter on a set  $I$  of cardinality  $\lambda^{<\theta} = \lambda$ , so can serve as  $D$  in the claim.

Let  $G_s : \mathcal{P}(s) \rightarrow |\mathcal{P}(s)|$  be one to one for  $s \in I$ .

By the normality of  $D$ , in  $(\theta, <)^I/D$ , the  $\theta$ -th element is  $f_0/D$  where  $f_0(s) = \min(\theta \setminus s)$ .

Now clause (b) holds for the function  $f_* \circ f_0$ , because  $\prod_{s \in I} (f_* \circ f_0)(s), <$  is isomorphic to  $(\lambda, <)$ , hence  $f_* \circ f_0/D$  is the  $\lambda$ -th member of  $(\theta, <)^I/D$ . As for clause (a) if  $g/D \in \theta^I/D$ ,  $\text{Rang}(g) \subseteq S_1$  and  $g <_D f_* \circ f_0$  then by the normality of  $D$ ,  $\prod_s g(s)/D$  has cardinality  $< \theta$ .

Note that  $f_* \circ f_0(s) = \min\{\gamma \in S_1 : \gamma > \sup(s \cap \theta)\}$ . Also clause (c) follows.

To prove clause (d) let  $f_2 \in I_\theta$  be  $f_2(s) = \min\{\gamma \in S_2 : \gamma > \sup(s \cap \theta)\}$ , so by  $f_2(s) = 2^{f_*(s \cap \theta)}$  when  $s \cap \theta \in E$  and easily  $\prod_{s \in I} f_2(s)/D$  is of cardinality  $\leq \theta^I = \theta^\lambda = 2^\lambda$ . In fact, it is of cardinality  $2^\lambda$  as exemplified by  $\langle f_{\mathcal{U}}/D : \mathcal{U} \subseteq \lambda \rangle$  where for  $\mathcal{U} \subseteq \lambda$  let  $f_{\mathcal{U}} : I \rightarrow \theta$  be  $f_{\mathcal{U}}(s) = G_s(\mathcal{U} \cap s)$ .  $\square_{2.19}$

**Claim 2.20.**  $M^I/D$  is locally  $(\mu, \lambda^+, \mathbb{L}_{\theta, \theta})$ -saturated when:

- $\boxplus$  (a)  $|T| = \theta$  and  $T$  is definably stable
- (b)  $D \in \text{ruf}_\theta(\lambda)$
- (c) if  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_T)$  then  $\mu \leq \text{lcr}(\text{spec}, D)$ .

*Proof.* Like 2.15.  $\square_{2.20}$

**Definition 2.21.** 1) Let  $T \subseteq \mathbb{L}_{\theta, \theta}(\tau_T)$  be complete. We say  $T$  has the global c.p. (negation: global n.c.p.) when for some pair  $(\bar{\varphi}, \bar{\partial})$  it has the global  $(\bar{\varphi}, \bar{\partial})$ -c.p., see below.

2)  $T$  has the global  $(\bar{\varphi}, \bar{\partial})$ -c.p. when for some  $S$  and  $\varepsilon$ :

- (a)  $S \subseteq \theta$  belongs to some normal ultrafilter on  $\theta$  and is a set of cardinals
- (b)  $\varepsilon < \theta$  and  $\bar{\varphi} = \langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_{\varphi_\alpha}) : \alpha < \theta \rangle$
- (c)  $\bar{\partial} = \langle \partial_\alpha : \alpha \in S \rangle$  and  $\partial_\alpha$  is a cardinal  $\in [\alpha, \theta)$
- (d) if  $\alpha \in S$  then  $\partial_\alpha \in \text{spec}(\bar{\varphi} \upharpoonright \alpha, T)$ , see Definition 2.8(3).

**Claim 2.22.** *If  $D$  is a normal ultrafilter on  $\theta$  and  $T$  has the global  $(\bar{\varphi}, \bar{\partial})$ -c.p.,  $S = \text{Dom}(\bar{\partial}) \in D$  and  $M$  is a model of  $T$  and  $\chi = \theta^\theta/D$  or just  $\chi = \Pi\bar{\partial}/D$  then  $N = M^\theta/D$  is not fully  $(\chi^+, \theta, \mathbb{L}_{\theta, \theta})$ -saturated.*

*Proof.* Let  $M \models T$  and for  $i \in S$  let  $\langle \varphi_{\xi(i, j)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i, j}) : j < \partial_i \rangle$  witness  $\partial_i \in \text{spec}(\bar{\varphi}|i, T)$  and  $\xi(i, j) < i$ . Let  $\partial'_\varepsilon$  be  $\partial_\varepsilon$  if  $\varepsilon \in S$  and 1 if  $\varepsilon \in \lambda \setminus S$ . We can fix  $\bar{f} = \langle f_\alpha : \alpha < \chi \rangle$  such that  $f_\alpha \in \prod_{\varepsilon < \theta} \partial'_\varepsilon$  and  $\bar{f}$  is a set of representatives for  $\prod_{i < \theta} \partial'_i/D$ .

For each  $\alpha < \chi$ , as  $D$  is a normal ultrafilter on  $\theta$  and  $i \in S \Rightarrow \xi(i, f_\alpha(i)) < i$  clearly for some  $\zeta(\alpha) < \theta$  we have  $S_\alpha := \{i < \theta : i \in S \text{ and } \xi(i, f_\alpha(i)) = \zeta(\alpha)\} \in D$  and let  $\bar{a}_\alpha^* \subseteq N$  be of length  $\ell g(\bar{y}_{\varphi_{\zeta(\alpha)}})$  such that  $\bar{a}_\alpha = \langle \bar{a}_{i, f_\alpha(i)} : i \in S_\alpha \rangle/D$  and let  $\Gamma = \{\varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha) : \alpha < \chi\}$ .

Of course,

(\*)<sub>1</sub>  $\Gamma$  is a set of  $\mathbb{L}_{\theta, \theta}(\tau_T)$ -formulas with parameters from  $N$

(\*)<sub>2</sub>  $\Gamma$  is  $(< \theta)$ -satisfiable  $M$ .

[Why? Let  $u \subseteq \chi$  have cardinality  $< \theta$ , hence  $\zeta(*) = \sup\{\zeta(\alpha) : \alpha \in u\}$  is  $< \theta$  and let  $S_* = \{i \in S : \text{if } \alpha \in u \text{ then } f_\alpha(i) = \zeta(\alpha) \text{ and } |u| < i\}$ . Clearly  $S_* \in D$  and if  $i \in S_*$  then  $\{\varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i, f_\alpha(i)}) : \alpha \in u\} \subseteq \{\varphi_{\xi(i, j)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i, j}) : j < \partial_i\}$  and<sup>3</sup> has cardinality  $< |i| < \partial_i$  hence is realized in  $M$ , so  $M \models (\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in u} \varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i, f_\alpha(i)})$ ,

hence  $N \models (\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in u} \varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha)$  so we are done.]

(\*)<sub>3</sub>  $\Gamma$  is not realized in  $N$ .

[Why? As in the proof of Case 2 of 2.13, without loss of generality  $\theta \subseteq M$ . Let  $\tau^* = \tau_T \cup \{P_\zeta, Q, <, R, F : \zeta < \theta\}$  where  $P_\zeta$  is a  $2 + \ell g(\bar{y}_{\varphi_\zeta})$ -place predicate,  $Q$  is unary,  $R$  is a  $1 + \varepsilon$  place predicate and  $F$  a unary function symbol.

For  $i \in S$  let  $M_i^+ = (M, Q^{M_i^+}, P_\zeta^{M_i^+}, <^{M_i^+}, R^{M_i^+}, F^{M_i^+})_{\zeta < \theta}$  where

- (\*)<sub>3.1</sub> •  $Q^{M_i^+} = \partial_i$
- $<^{M_i^+}$  the order on  $\partial_i$
- $P_\zeta^{M_i^+} = \{\langle \zeta, j \rangle^{\wedge} \bar{a}_{i, j} : j < \partial_i, \xi(i, j) = \zeta\}$
- $R^{M_i^+} = \{\langle j \rangle^{\wedge} \bar{b} : j < \partial_i \text{ and } \ell g(\bar{b}) = \varepsilon \text{ and } M \models \varphi_{\xi(i, j)}[\bar{b}, \bar{a}_{i, j}]\}$
- $F^{M_i^+}(j) = \xi(i, j) < i$ .

Let  $N^+ = \prod_{i \in S} M_i^+/D$ , so  $N = N^+ \upharpoonright \tau_T$ , let  $\mathbf{i} = \langle i : i \in S \rangle/D \in N^+$  and  $\partial = \langle \partial_i : i \in S \rangle/D \in N^+$

(\*)<sub>3.2</sub> in  $N^+$  there is no  $\bar{b} \in {}^\varepsilon(N^+)$  such that for every  $j \in Q^{N^+}$ ,  $N^+ \models "j < \partial \rightarrow R[j, \bar{b}]"$

(\*)<sub>3.3</sub> in  $N^+$  if  $j \in Q^{N^+}$  and  $F^{N^+}(j) = \zeta < \theta$  then  $N^+ \models (\forall \bar{x}_{[\varepsilon]})(\forall \bar{y})[P_\zeta(j, \zeta, \bar{y}) \rightarrow R(j, \bar{x}_{[\varepsilon]}) \equiv \varphi_\zeta(\bar{x}_{[\varepsilon]}, \bar{y})]$ .

Let

(\*)<sub>3.4</sub>  $\Gamma = \{\varphi_\zeta(\bar{x}_{[\varepsilon]}, \bar{a}) : \text{for some } j \in Q^{N^+}, \zeta = F^{N^+}(j) \text{ we have } N^+ \models "P_\zeta(j, \zeta, \bar{a})"\}$ .

<sup>3</sup>The  $\leq \partial_i$  is for technical reasons anyhow,  $\partial_1 = |\partial_1| + 1$ .

Together

- (\*)<sub>3.5</sub>  $\Gamma$  is a set of  $\chi, L_{\theta, \tau}(\tau_T)$ -formulas with parameters from  $N$ , ( $< \theta$ )-satisfiable in  $N$  but not realized in  $N$  so we are done.

□<sub>2.22</sub>

**Claim 2.23.** *There are a vocabulary  $\tau, |\tau| \leq \theta$  and a complete  $T \subseteq \mathbb{L}_{\theta, \theta}(\tau)$  which have  $\theta$ -n.c.p. but has the global c.p.*

*Proof.* Let  $\partial_i$  be an infinite cardinal  $\in [i, \theta]$ .

We choose a model  $M$  as follows:

- (a) its universe is  $\theta \times \theta$
- (b)  $E^M = \{((i, j_1), (i, j_2)) : i < \theta \text{ and } j_1, j_2 < \theta\}$ , an equivalence relation
- (c)  $P_\zeta^M \subseteq |M|$  for  $\zeta < \theta$
- (d) letting  $a_i = (i, 0)$ ,  $A_i = a_i/E^M$  for  $i < \theta$ , for every  $\eta \in {}^i 2$  the following are equivalent:
  - ( $\alpha$ ) there are  $\theta_i$  elements  $a \in A_i$  such that  $(\forall \zeta < i)(a \in P_\zeta^M \equiv \eta(\zeta) = 1)$
  - ( $\beta$ ) the set  $\{a \in A_i : \text{if } \zeta < i \text{ then } a \in P_\zeta^M \Leftrightarrow \eta(\zeta) = 1\}$  has cardinality  $\neq \partial_i$
  - ( $\gamma$ ) the set  $\{j < i : \eta(j) = 1\}$  has cardinality  $< 1 + |i|$ .

We shall check that  $T := \text{Th}_{\mathbb{L}_{\theta, \theta}(\tau)}(M)$  is as required.

Let  $A'_i = \{a \in A_i : \text{if } i < i \text{ then } a \in P_i^M\}$ ; it is a subset of  $A_i$  of cardinality exactly  $\partial_i$  by clause (d) above

⊞<sub>1</sub>  $T$  has global  $\theta$ -c.p.

Why? Let  $\varepsilon = 1, \bar{y} = \langle y_0, y_1 \rangle$  and  $\varphi_i = \varphi_i(x, \bar{y}) = xEy_0 \wedge P_i(x) \wedge x \neq y_1$  for  $i < \theta$  and let  $\bar{\varphi} = \langle \varphi_i : i < \theta \rangle$ .

For  $i < \theta$  let  $\Gamma_i = \{\varphi_j(x, \langle a_i, b \rangle) : b \in A_i\}$

- $\Gamma_i$  is formally is as required for witnessing  $\partial_i \in \text{spec}(\bar{\varphi} \upharpoonright i, T)$  in particular  $|\Gamma_i| = \partial_i$
- $\Gamma_i$  is not realized.

[Why? As every  $\{xEa_i \wedge x \neq b \wedge P_\zeta(x) : b \in A'_i \text{ and } \zeta < i\}$  is not realized.]

- if  $\Gamma \subseteq \Gamma_i$  has cardinality  $< \partial_i$  then  $\Gamma$  is realized.

[Why? As all but  $< \partial_i$  members of  $A'_i$  realizes  $\Gamma$ .]

So ⊞<sub>1</sub> holds indeed.

⊞<sub>2</sub>  $T$  has the  $\theta$ -n.c.p.

[Why? Let  $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$  and so for some  $\kappa < \theta, \varphi$  belongs to  $\mathbb{L}_{\theta, \theta}(\{\theta, P_\zeta : \zeta < \kappa\})$ , hence  $M$  satisfies:

- if  $a \in M, a \notin a_j/E^M$  for  $j < \kappa^+$  then for any  $\eta \in {}^2$  the set  $\{b : b \in a/E^M \text{ and } \zeta < \kappa \Rightarrow b \in P_\zeta^M \Leftrightarrow \eta(\zeta) = 1\}$  has cardinality  $\theta$ .

The rest should be clear.

⊞<sub>3</sub>  $T$  is 1-stable.

[Why? Obvious.]

Together we are done.  $\square_{2.23}$

**Theorem 2.24.** *Assume  $T$  is complete of cardinality  $\theta$  and  $T$  is definably stable with global  $\theta$ -n.c.p. and  $\lambda = \lambda^{<\theta}$ .*

1)  *$T$  is  $\triangleleft_{\lambda,\kappa}^{\text{ful}}$ -minimal.*

2) *Moreover, if  $D \in \text{uf}_{\lambda,\theta}(I)$  and  $\theta^I/D > \lambda$  and  $M$  is a model of  $T$  then  $M^I/D$  is fully  $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$ -saturated.*

*Proof.* 1) By part (2).

2) As  $T$  is definably stable we can use 1.8(2) and as  $T$  has  $\theta$ -n.c.p. we can use 2.8, 2.9.

Let  $M \models T$  and  $N = M^I/D$ , let  $A \subseteq N$ ,  $|A| \leq \lambda$  and  $p \in \mathbf{S}^\varepsilon(A, N)$  and we shall prove that  $p_0(\bar{x}_{[\varepsilon]})$  is realized; by 2.3 without loss of generality  $M$  is  $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$ -saturated. Let  $\Phi_{T,\varepsilon}$  be listed as  $\langle \varphi_i(\bar{x}_{[\varepsilon]}, \bar{y}_{\zeta(i)}) : i < \theta \rangle$ . Let  $p_1(\bar{x}_{[\varepsilon]}) \in \mathbf{S}^\varepsilon(N)$  extends  $p_0(\bar{x}_{[\varepsilon]})$  and for each  $i < \theta$  let  $\psi_i = \psi_i(\bar{y}_{\zeta(i)}, \bar{c}_i)$  be a formula from  $\mathbb{L}_{\theta,\theta}(\tau_T)$  with parameters from  $N$  defining  $p_1(\bar{x}_{[\varepsilon]}) \upharpoonright \varphi_i$  and let  $\bar{c}_\zeta = \langle \bar{c}_{\zeta,s} : s \in I \rangle/D$ .

As  $D$  is a  $(\lambda, \theta)$ -regular ultrafilter by 0.14(2) there is  $\bar{A} = \langle A_s : s \in I \rangle$ ,  $A_s \in [M_s]^{<\theta}$  which is non-empty and  $A = \{f_\alpha/D : \alpha < \lambda\}$  and  $\alpha < \lambda \Rightarrow f_\alpha \in \prod_{s \in I} A_s$  and for  $i \leq \theta$  let  $\Delta_i = \{\varphi_j(\bar{x}_{[\varepsilon]}, \bar{y}_{\zeta(\bar{y})}) : j < i\}$  and let  $p_{s,i}(\bar{x}_{[\varepsilon]}) = \{\varphi_j(\bar{x}_{[\varepsilon]}, \bar{c}) : j < i, \bar{c} \in A_s, M \models \psi_j(\bar{b}, \bar{c}_{j,s})\}$ .

For each  $i < \theta$  let  $\partial_i = \sup(\text{spec}(\Delta_i, T))$ , see 2.8(3) so  $\partial_i < \theta$  and let  $I_i = \{s \in M : \text{there is } p \in \mathbf{S}_{\Delta_i}^\varepsilon(A_\alpha) \text{ such that } \psi_j(\bar{y}_{[\zeta(j)]}, \bar{c}_{j,s}) \text{ define } p \upharpoonright \varphi_j \text{ for each } j < i\}$ . Clearly  $I_i \in D$  is decreasing with  $i$ . Let  $I'_\theta = \cap \{I_j : j < \theta\}$  and for  $i < \theta$  let  $I'_i = \cap \{I_j : j < i\} \setminus I_i$  for  $i < \theta$ ; so  $I'_0 = I \setminus I_0$  and  $\langle I'_i : i < \theta \rangle$  is a partition of  $I \setminus I'_\theta$  to  $\theta$  sets  $= \emptyset \pmod D$ .

If  $I'_\theta \in D$ , recall that  $M$  is  $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$ -saturated, hence we can find  $f \in {}^I M$  such that  $s \in I'_\theta \Rightarrow f(s)$  realizes  $p_{s,\theta}$ , clearly  $f/D$  realizes  $p$  in  $N$  so we are done; hence without loss of generality  $I'_\theta = \emptyset$ .

We can find  $\mathbf{h} : I \rightarrow \theta$  such that  $s \in I'_i \Rightarrow \mathbf{h}(s) = i$  clearly.

Let  $\mathbf{h}_* \in {}^I \theta$  be such that  $(\theta, \mathbf{h}_*/D)$  is the  $\theta$ -th member of  $(Q, <)^I$  and without loss of generality  $\mathbf{h}_* \leq \mathbf{h}$ .

Case 1:  $\mathbf{h}_* <_D \mathbf{h}$ .

In this case we can prove that  $p_0(\bar{x}_{[\varepsilon]})$  is realized in  $N$ .

Case 2: Not Case 1.

In this case we can prove that  $T$  has global  $\theta$ -c.p., contradicting an assumption.  $\square_{2.24}$

**Conclusion 2.25.** *Assume  $\lambda \geq 2^\theta$ ,  $T$  is a complete  $\mathbb{L}_{\theta,\theta}(\tau_T)$ -theory of cardinality  $\theta$ . Then  $T$  is  $\triangleleft_{\lambda,\theta}^{\text{ful}}$ -minimal iff  $T$  is definably stable and globally  $\theta$ -n.c.p.*

*Proof.* Like the proof of 2.16 by using 2.24 instead of 2.13.  $\square_{2.25}$

### § 3. ON $\mathbb{L}_{<\theta}^1$ EXTRAPOLATING $\mathbb{L}_{\theta, \aleph_0}$ AND $\mathbb{L}_{\theta, \theta}$

In [Sh:797], a logic  $\mathbb{L}_{<\kappa}^1$  has introduced (assume  $\kappa$  is strongly inaccessible for transparency), and is proved to be stronger than  $\mathbb{L}_{\kappa, \aleph_0}$  but weaker than  $\mathbb{L}_{\kappa, \kappa}$ , has interpolation and a characterization, well ordering not definable in it; and it is the maximal logic with some such properties.

For  $\kappa = \theta$ , we give a characterization of when two models are  $\mathbb{L}_{<\theta}^1$ -equivalent giving an additional evidence for the logic naturality.

Recall [Sh:797, 2.11=a18] is, it says

**Claim 3.1.** *Assume  $|\tau| \leq \mu, \alpha_* < \mu^+, M_n$  is a  $\tau$ -model and  $M_n \prec_{\mu^+} M_{n+1}^\ell$  for  $n < \omega$  and  $M = \cup\{M_n : n < \omega\}$ . Then  $M_0, M_\omega$  are  $\mathbb{L}_{\leq \mu}^1$ -equivalent.*

**Theorem 3.2.** *Assuming  $M_1, M_2$  are  $\tau$ -models of cardinality  $\leq \kappa$  and  $\text{arity}(\tau) = \aleph_0$ , i.e. the arity of every symbol from  $\tau$  is finite then the following are equivalent:*

- (a)  $M_1, M_2$  are  $\mathbb{L}_{<\theta}^1$ -equivalent
- (b) if  $\kappa \geq \|M_1\| + \|M_2\| + \theta$  then there are  $(\theta, \theta)$ -u.f.l.p.  $\mathbf{x}_n = (I, D, \mathcal{E}_n), \mathcal{E}_n \subseteq \mathcal{E}_{n+1}$  for  $n < \omega$  and we let  $\mathcal{E} = \cup\{\mathcal{E}_n : n < \omega\}$  such that  $(M_1)_D^I|_{\mathcal{E}}$  is isomorphic to  $(M_2)_0^I|_{\mathcal{E}}$  and  $|I| \leq 2^\kappa$
- (c)  $(M_1, M_2)$  have isomorphic  $\theta$ -complete  $\omega$ -iterated ultrapowers that is we can find  $D_n \in \text{uf}_\theta(I_n)$  for  $n < \omega$  such that
  - $(*)_{M_1, M_2, \langle I_n, D_n : n < \omega \rangle}$  if  $M_0^\ell = M_\ell, M_n^\ell \prec_\theta (M_n^\ell)^{I_n}/D_n = M_{n+1}^\ell$  for  $n < \omega$  and  $M_\omega^\ell = \cup\{M_k^\ell : k < \omega\}$  for  $\ell = 1, 2$  and  $n < \omega$  then  $M_\omega^1 \cong M_\omega^2$
- (d) if  $D_n \in \text{uf}_{\lambda_n, \theta}(I_n)$  and  $\lambda_{n+1} \geq 2^{|I_n|} + \|M_1\| + \|M_2\|$  then the sequence  $\langle (I_n, D_n) : n < \omega \rangle$  is as required in clause (c)
- (e) if  $(I, D, \mathcal{E})$  is a u.f.l.p.,  $\mathcal{E} = \{E_n : n < \omega\}, E_{n+1}$  refines  $E_n, 2^{|I/E_n|} \leq \lambda_{n+1}, D/E_n$  is a  $(\lambda_n, \theta)$ -regular hence  $\theta$ -complete ultrafilter,  $\bar{w}$  is a niceness witness, see below, then  $\text{u.f.l.p.}_{\mathbf{x}}(M_1) \cong \text{u.f.l.p.}_{\mathbf{x}}(M_2)$  where
  - $\oplus \bar{w}$  is a niceness witness for  $(I, D, \bar{E})$ , where  $\bar{E} = \langle E_n : n < \omega \rangle$  when  $I, D, \bar{E}$  are above and:
    - (a)  $\bar{w} = \langle w_{s,n}, \gamma_{n,s} : s \in I, n < \omega \rangle$
    - (b)  $w_{s(*)} \subseteq \lambda_n$  has cardinality  $< \theta$
    - (c)  $|w_{s,n}| \geq |w_{s,n+1}|$  and  $\gamma_{s,n} > \gamma_{s,n+1} \vee (\gamma_{s,n+1} = 0)$
    - (d)  $w_{s,n} = \emptyset \Rightarrow \bigwedge_{k \geq n} \gamma_{s,k} = 0$
    - (e) if  $n < \omega, u \in [\lambda_n]^{<\theta}$  then  $\{s \in I : u \subseteq w_{n,s}\} \in D$
    - (f)  $w_{s,n} = w_{t,n}$  and  $\gamma_{s,n} = \gamma_{t,n}$  when  $sE_n t$
    - (g)  $w_{s,0}$  is infinite for every  $s \in I$ , for simplicity.

*Proof.* Clause (b)  $\Rightarrow$  Clause (a):

So let  $I, D, \mathcal{E}_n (n < \omega)$  be as in clause (b) and  $\mathcal{E} = \cup\{\mathcal{E}_n : n < \omega\}$ . By the transitivity of being  $\mathbb{L}_{<\theta}^1$ -equivalent, clearly clause (a) follows from:

$\boxplus_1$  for every model  $N$  the models  $N, N_D^I|_{\mathcal{E}}$  are  $\mathbb{L}_{<\theta}^1$ -equivalent.

[Why?  $N_n = N_D^I|_{\mathcal{E}_n}$  for  $n < \omega$  and  $N_\omega = \cup\{N_n : n < \omega\}$ . So by 0.20 we have  $N \equiv_{\mathbb{L}_{\theta,\theta}} N_0$  and  $N_n \prec_{\mathbb{L}_{\theta,\theta}} N_{n+1}$ . Hence by 3.1, that is the “Crucial Claim” [Sh:797, 2.11=a18] we have  $N_n \equiv_{\mathbb{L}_{<\theta}^1} N_\omega$  hence  $N \equiv_{\mathbb{L}_{<\theta}^1} N_\omega$ .]

Clause (c)  $\Rightarrow$  Clause (b):

Let  $I = \prod_{n < \omega} I_n, E_n = \{(\eta, \nu) : \eta, \nu \in I \text{ and } \eta \upharpoonright n = \nu \upharpoonright n\}$  and  $D = \{X : \text{for some } n, (\forall^{D_n} i_n \in I_n)(\forall^{D_{n-1}} i_{n-1} \in I_{n-1}) \dots (\forall^{D_0} i_0 \in I_0)(\forall \eta)[\eta \in I \wedge \bigwedge_{\ell \leq n} \eta(\ell) = i_n \rightarrow \eta \in X]\}$ . Now let  $M_\omega^\ell \equiv (M_\ell)_D^I|\{E_n : n < \omega\}$ .

So  $(M_\ell)_D^I|\{E_n : n < \omega\}$  is isomorphic to  $M_\omega^\ell$  for  $\ell = 1, 2$ , so recalling  $M_\omega^1 \cong M_\omega^2$  by the present assumptions, the models  $(M_\ell)_D^I|\{E_n : n < \omega\}$  for  $\ell = 1, 2$  are isomorphic, so we are done except that possibly  $|I| > 2^\kappa$ . But by the downward LST argument we can finish.

Clause (d)  $\Rightarrow$  Clause (c):

Clause (d) is obviously stronger because if  $\lambda_0 = \|M_1\| + \|M_2\|, \lambda_{n+1} = 2^{\lambda_n}$  then letting  $I_n = \lambda_n$  there is  $D_n \in \text{uf}_{\lambda_n, \theta}(I_n)$ .

Clause (e)  $\Rightarrow$  Clause (d):

Let  $\langle (I_n, D_n, \lambda_n) : n < \omega \rangle$  be as in the assumption of clause (d).

We define  $I = \prod_n I_n, E_n = \{(\eta, \nu) : \eta, \nu \in I, \eta \upharpoonright (n+1) = \nu \upharpoonright (n+1)\}$  and define  $D$  as in the proof of (c)  $\Rightarrow$  (b) above and we define  $\bar{w} = \langle w_{\eta,n} : \eta \in I, n < \omega \rangle$  as follows: choose  $\langle u_s^n : s \in I_n \rangle$  which witness  $D_n$  is  $(\lambda_n, \theta)$ -regular, i.e.  $u_s^n \in [\lambda_n]^{<\theta}$  and  $(\forall \alpha < \lambda_n)[\{s \in I_n : \alpha \in u_s^n\} \in D_n]$ .

Let  $w_{\eta,n}$  be  $u_{\eta(n)}^n$  if  $\langle \text{otp}(u_{\eta(\ell)}^n) : \ell \leq n \rangle$  is decreasing and  $\emptyset$  otherwise. Now check that the assumptions of clause (e) holds, hence its conclusion and we are done as in the proof of (c)  $\Rightarrow$  (b).

Clause (a)  $\Rightarrow$  Clause (e):

So assume that clause (a) holds that is  $M_1, M_2$  are  $\mathbb{L}_{<\theta}^1$ -equivalent and  $I, D, \mathcal{E}, \langle E_n : n < \omega \rangle$  and  $\bar{w}$  are as in the assumption of clause (e), and we should prove that its conclusion, that is,  $\text{u.f.l.p.}_\mathbf{x}(M_1) \cong \text{u.f.l.p.}_\mathbf{x}(M_2)$ .

For every  $\tau_* \subseteq \tau$  of cardinality  $< \theta$  and  $\mu < \theta$ , we know that  $M_1 \upharpoonright \tau_*, M_2 \upharpoonright \tau_*$  are  $\mathbb{L}_{\leq \mu}^1$ -equivalent, hence for every  $\alpha < \mu^+$  there is a finite sequence  $\langle N_{\tau_*, \mu, \alpha, k} : k \leq \mathbf{k}(\tau_*, \mu, \alpha) \rangle$  such that (see [Sh:797, 2.1=a8(?)]:

- (\*)<sub>1</sub> (a)  $N_{\tau_*, \mu, \alpha, 0} = M_1 \upharpoonright \tau_*$
- (b)  $N_{\tau_*, \mu, \alpha, \mathbf{k}(\tau_*, \mu, \alpha)} = M_2 \upharpoonright \tau_*$
- (c) in the game  $\mathfrak{D}_{\tau_*, \mu, \alpha}[N_{\tau_*, \mu, \alpha, k}, N_{\tau_*, \mu, \alpha, k+1}]$  the ISO player has a winning strategy for each  $k < \mathbf{k}(\tau_*, \mu, \alpha)$ , but we stipulate a play to have  $\omega$  moves, stipulating they continue to choose in the move when one side wins
- (\*)<sub>2</sub> without loss of generality  $\|N_{\tau_*, \mu, \alpha, k}\| \leq \lambda_0$  of  $k \in \{1, \dots, \mathbf{k}(\tau_*, \mu, \alpha) - 1\}$  (even  $< \theta$ ).

By monotonicity we can (for transparency) assume:

- (\*)<sub>3</sub> (a) above  $\mathbf{k}(\tau_*, \mu, \alpha) = \mathbf{k}$
- (b)  $\tau$  have only predicates

- (\*)<sub>4</sub> (a)  $\langle P_\alpha : \alpha < |\tau| \rangle$  list the predicates of  $\tau$ , note that necessarily  $|\tau| \leq \lambda_0$
- (b) for  $t \in I$  let  $\tau_t = \{P_\alpha : \alpha \in w_{t,0} \cap |\tau|\}$
- (\*)<sub>5</sub> let  $N_{s,k}^\ell := N_{\tau_s, |w_{s,0}|, \gamma_{s,0}+1, k}^\ell$  for  $s \in I$ .

Let  $\langle f_\alpha : \alpha < 2^{\lambda_n} \rangle$  list the members  $f$  of  $\prod_{s \in I} N_{s,k}^\ell$  such that  $E_n$  refines  $\text{eq}(f)$ , so  $f_{k,n,\alpha} = \langle f_{k,n,\alpha}(\eta) : \eta \in I \rangle$  but  $\eta \in I \wedge \nu \in I \wedge \eta E_n \nu \Rightarrow f_{k,n,\alpha}(\eta) = f_{k,n,\alpha}(\nu)$ .

- (\*)<sub>6</sub> (a) for  $t \in I$  and  $k < \mathbf{k}$  let  $\mathcal{D}_{t,k}$  be the game  $\mathcal{D}_{\tau_t, |w_{t,0}|, \gamma_{t,0}+1} [N_{t,k}^1, N_{t,k}^2]$  and
- (b) let  $\mathbf{st}_{t,\ell}$  be a winning strategy for the ISO player in  $\mathcal{D}_{t,\ell}$
- (c) if  $t_1 E_0 t_2$  then  $\langle N_{t_\iota, k}^\ell : k \leq \mathbf{k}, \iota \in \{1, 2\} \rangle$  are the same for  $\iota = 1, 2$ , moreover  $(\mathcal{D}_{t_1} = \mathcal{D}_{t_2})$  and  $\mathbf{st}_{t_1, k} = \mathbf{st}_{t_2, k}$ .

Now by induction on  $n$  we choose  $\mathbf{s}_{t,k,n}$  such that

- (\*)<sub>7</sub> (a)  $\mathbf{s}_{t,k,n}$  is a state of the game  $\mathcal{D}_{t,k}$
- (b)  $\langle \mathbf{s}_{t,k,\ell} : \ell \leq n \rangle$  is an initial segment of a game of  $\mathcal{D}_{t,k}$  in which the ISO player uses the strategy  $\mathbf{st}_{t,k}$
- (c) if  $t_1 E_n t_2$  then  $\mathbf{s}_{t_1, k, n} = \mathbf{s}_{t_2, k, n}$
- (d)  $\beta_{\mathbf{s}_{t,k,n}} = \text{otp}(w_{t,n})$ , see [Sh:797, 2.1=a8]
- (e) if  $n = \iota \bmod 2$  and  $\iota \in \{1, 2\}$  then  $A_{\mathbf{s}_{t,k,n}}^\iota \supseteq \{f_{k,m,\alpha}(s) : m < n \text{ and } \alpha \in w_{s,m}\}$
- (\*)<sub>8</sub> we can carry the induction on  $n$ .

[Why? Straight.]

- (\*)<sub>9</sub> for each  $k < \mathbf{k}, n < \omega, t \in I$  we define  $h_{s,k,n}$ , a partial function from  $N_{s,k}$  to  $N_{s,k+1}$  by  $h_{s,k,n}(a_1) = a_2$  iff for some  $m \leq n, w_{s,m} \neq \emptyset$  and  $g_{\mathbf{s}_{t,k,m}}(a_1) = a_2$
- $\boxplus_1$  for each  $t \in I, k < \mathbf{k}$  and  $n < \omega, h_{s,k,n}$  is a partial one-to-one function from  $N_{s,k}$  to  $N_{s,k+1}$ , non-empty increasing with  $n$
- $\boxplus_2$  let  $Y_{k,n} = \{(f_1, f_2) : f_1 \in \prod_{s \in I} \text{Dom}(h_{s,k,n}), \text{Dom}(f_2) = I \text{ and } s \in \text{Dom}(f_1) \Rightarrow f_2(s) = h_{s,k,n}(f_1(s))\}$
- $\boxplus_3$   $\mathbf{f}_{k,n} = \{(f_1/D, f_2/D) : (f_1, f_2) \in Y_{k,n}\}$  is a partial isomorphism from  $M_1^I \upharpoonright \{f/D : f \in \prod_s N_{s,k} \text{ and } f \text{ respects } E_n\}$  to  $M_2^I \upharpoonright \{f/D : f \in \prod_s N_{s,k+1} \text{ and } f \text{ respects } E_n\}$
- $\boxplus_4$  (a)  $\mathbf{f}_{k,n} \subseteq \mathbf{f}_{k,n+1}$
- (b) if  $\iota \in \{1, 2\}$  and  $f_\iota \in \prod_s N_{s,k}$  and  $\text{eq}(f)$  is refined by  $E_n$  then for some  $n_1 > n$  and  $f_{3-\iota} \in \prod_s N_{s,k+1}$  the pair  $(f_1/D, f_2/D)$  belongs to  $\mathbf{f}_{k,n_1}$ .

[Why? For some  $\alpha, f_\iota = f_{n,\alpha}^\iota$ , hence for every  $t, f_\iota(t) \in A_{\mathbf{s}_{t,n}}^\iota$ . We use the “delaying function”,  $h_{\mathbf{s}_{t,n}}(f_\iota(t)) < \omega$  and for some  $k$  the set  $\{t \in I : h_{\mathbf{s}_{t,n}}(f_\iota(t)) \leq k\}$  which respects  $E_n$  belongs to  $D$ .]

Putting together



(\*)<sub>10</sub>  $\mathbf{f}_k = \bigcup_n \mathbf{f}_{k,n}$  is an isomorphism from  $(\prod_s N_{k,s})_D | \mathcal{E}$  onto  $(\prod_s N_{k+1,s})_D | \mathcal{E}$ .

Hence

(\*)<sub>11</sub>  $f_{\mathbf{k}-1} \circ \dots \circ \mathbf{f}_0$  is an isomorphism from  $(M_1)_D^I | \mathcal{E}$  onto  $(M_2)_D^I | \mathcal{E}$ .

So we are done. □<sub>3.3</sub>

**Discussion 3.3.** 1) So for our  $\theta$ , we get another characterization of  $\mathbb{L}_{<\theta}^1$ .

2) Probably we can conclude that “distance 2” suffice in [Sh:797, §2].

3) Deal with universal homogeneous  $(\theta, \sigma)$ -u.f.l.p.  $\mathbf{x}$ , at least for  $\sigma = \aleph_0$ , using Definition 0.17.

4) The logic with EM models.

## § 4. CONCLUDING REMARKS

## § 4(A). Consequences of Categoricity.

**Theorem 4.1.** *Assume  $T$  is definably unstable.  
 $T$  is not categorical in  $\lambda$  when  $\lambda \geq 2^\theta$ .*

So for this subsection

**Hypothesis 4.2.**  $T$  is (complete theory in  $\mathbb{L}_{\theta,\theta}(\tau_T)$  and is) definable unstable as witnessed by  $\varphi = \varphi(\bar{x}_\varepsilon, \bar{y}_\zeta)$ .

**Claim 4.3.** *For every  $\lambda$  there are  $M$  and  $\bar{\mathbf{b}}$  such that:*

- (a)  $M$  a model of  $T$
- (b)  $\bar{\mathbf{b}} = \langle (\bar{b}_u^0, \bar{b}_u^1, \bar{a}_u) : u \in \mathcal{P} \rangle$
- (c)  $\mathcal{P}$  is a family of subsets of  $M$
- (d)  $\mathcal{P}$  is  $\lambda$ -directed
- (e) every  $a \in M$  belongs to some  $u \in \mathcal{P}$
- (f)  $\{\varphi(\bar{x}_\varepsilon, \bar{b}_u^1) \wedge \neg\varphi(\bar{x}_\varepsilon, \bar{b}_u^0) : u \in \mathcal{P}\}$  is  $(< \theta)$ -satisfiable in  $M$
- (g)  $\bar{b}_u^1, \bar{b}_u^2$  realize the same  $\mathbb{L}_{\theta,\theta}$ -type over  $u$
- (h)  $\bar{a}_u \in {}^\varepsilon M$  realizes  $\{\varphi(\bar{x}_\varepsilon, \bar{b}_u^1) \wedge \neg\varphi(\bar{x}_\varepsilon, \bar{b}_u^0) : v \subseteq u, \bar{b}_v^0 \wedge \bar{b}_v^1 \text{ is } \subseteq u\}$ .

*Proof.* Let  $M_*$  be a model of  $T$  such that  $p_* \in \mathbf{S}_\varphi^\varepsilon(M_*)$  is not definable. Clearly  $\|M\| \geq \theta$ , now without loss of generality  $\|M_*\| \geq |\mathbb{L}_{\theta,\theta}(\tau_T)|$  as we can use an ultrapower of  $M$  and let  $A_* \subseteq M$  have cardinality  $|\mathbb{L}_{\theta,\theta}(\tau_T)|$  and let  $\langle \vartheta_a(\bar{y}_\zeta, \bar{z}_a) : a \in A_* \rangle$  list the formulas from  $\mathbb{L}_{\theta,\theta}(\tau_T)$ .

Let  $\mathcal{P}_* = [M_*]^{<\theta}$  and for each  $u \in \mathcal{P}_*$  clearly there are  $\bar{b}_u^0, \bar{b}_u^1 \in {}^\zeta(M_*)$  such that

- (\*)<sub>u</sub> (a) if  $a \in u$  and  $\bar{c} \in {}^{\ell g(\bar{z}_a)}u$  then  $M_* \models \vartheta_a[\bar{b}_u^0, \bar{c}] \equiv \vartheta_a[\bar{b}_u^1, \bar{c}]$
- (b)  $\varphi(\bar{x}_\varepsilon, \bar{b}_u^1), \neg\varphi(\bar{x}_\varepsilon, \bar{b}_u^0) \in p_*(\bar{x}_\varepsilon)$ .

[Why? Because the set of  $\{(a, \bar{c}) : a \in u, \bar{c} \in {}^{\ell g(\bar{z}_a)}u\}$  has cardinality  $< |u| \cdot (|u|^{|\bar{u}|}) < \theta$  for every  $u \in \mathcal{P}_*$ .]

Let  $\bar{\mathbf{b}}_* = \langle (\bar{b}_u^0, \bar{b}_u^1) : u \in \mathcal{P}_* \rangle$ .

Let  $\chi$  be large enough and let  $I$  be such that  $|I| = |I|^{<\theta} > \lambda$ , let  $D \in \text{uf}_{\lambda,\theta}(I)$ ,  $\mathfrak{B}_0 = (\mathcal{H}(\chi), \in, M_*, \bar{\mathbf{b}}_*, \mathcal{P}_*, A_*, \theta, T)$  and let  $\mathfrak{B}_1 = \mathfrak{B}_0^I/D$  and  $\mathbf{j} : \mathfrak{B}_0 \rightarrow \mathfrak{B}_1$  be the canonical  $\prec_{\mathbb{L}_{\theta,\theta}}$ -embedding. Let  $M = M_*^{\mathfrak{B}_1}$  so clearly a model of  $T$ . Let  $\mathcal{P} = \{\{a : M \models "a \in u" : \mathfrak{B}_1 \models "u \in \mathcal{P}_*" \text{ and } a \in A_* \Rightarrow M \models "\mathbf{j}(a) \in u"\}\}$ . Now check, the  $\lambda^+$ -directed holds by 0.14(2).  $\square_{4.3}$

We can strengthen

**Claim 4.4.** *In 4.3 we can add:*

- (i) for some type  $q = q(\bar{y}_\varepsilon, \bar{x}_\zeta^0, \bar{x}_\zeta^1) \in \mathbf{S}^{\varepsilon+\zeta+\zeta}(M)$  we have  $u \in \mathcal{P} \Rightarrow \text{tp}(\bar{a}_u \wedge \bar{b}_u^0 \wedge \bar{b}_u^1, u, M) = q \upharpoonright u$ .

*Proof.* In the proof of 4.3 let  $E$  be a  $\theta$ -complete filter on  $\mathcal{P}$  such that  $u \in \mathcal{P} \Rightarrow \{v \in \mathcal{P} : u \subseteq v\} \in E$ . Let  $q = \text{Av}(\langle \bar{a}_u \wedge \bar{b}_u^0 \wedge \bar{b}_u^1 : u \in \mathcal{P} \rangle, E, M)$ , that is  $q = \{\varphi(\bar{y}_\varepsilon, \bar{x}_\varepsilon^0, \bar{x}_\varepsilon^1, \bar{c}) : \varphi \in \mathbb{L}_{\theta, \theta}(\tau_T), \bar{c} \in {}^\theta M \text{ and } \{u \in \mathcal{P} : M \models \varphi[\bar{a}_u, \bar{b}_u^0, \bar{b}_u^1, \bar{c}]\} \text{ belongs to } E\}$ , hence  $q \in \mathbf{S}^{\varepsilon+\zeta+\zeta}(M)$ . So there is  $F : \mathcal{P} \rightarrow \mathcal{P}$  such that  $u \subseteq F(u)$  and  $\text{tp}_{\mathbb{L}_{\theta, \theta}}(\bar{a}_u \wedge \bar{b}_u^0 \wedge \bar{b}_u^1, u, M) = q \upharpoonright u$ , renaming that is replacing  $(\bar{a}_u, \bar{b}_u^0, \bar{b}_u^1)$  by  $(\bar{a}_{F(u)}, \bar{b}_{F(u)}^0, \bar{b}_{F(u)}^1)$  we are done.  $\square_{4.4}$

**Definition 4.5.** Let  $M \models T$ .

1) We define  $\text{inc}_\varphi(M)$  as the set of regular  $\partial$  such that:

- (\*) $_{M, \varphi, \partial}^1$  there is a sequence  $\langle (M_i, \bar{b}_i^0, \bar{b}_i^1) : i < \partial \rangle$  such that
  - (a)  $\langle M_i : i < \partial \rangle$  is  $\prec_{\mathbb{L}_{\theta, \theta}}$ -increasing with union  $M$
  - (b)  $\{\varphi(\bar{x}_\varepsilon, \bar{b}_i^1) \wedge \neg \varphi(\bar{x}_\varepsilon, \bar{b}_i^0) : i < \partial\}$  is included in some  $p_* \in \mathbf{S}^\varepsilon(M)$
  - (c)  $\text{tp}_{\mathbb{L}_{\theta, \theta}}(\bar{b}_i^0, M_i, M) = \text{tp}_{\mathbb{L}_{\theta, \theta}}(\bar{b}_i^1, M_i, M)$
  - (d)  $\langle \text{tp}_{\mathbb{L}_{\theta, \theta}}(\bar{b}_i^0 \wedge \bar{b}_i^1, M_i, M) : i < \partial \rangle$  is increasing, call the union  $q_* \in \mathbf{S}^{\zeta+\zeta}(M)$ .

2) We define  $\text{comp}_\varphi(M)$  as the set of regular  $\partial$  such that

- (\*) $_{M, \varphi, \partial}^2$  if  $\bar{\mathbf{b}} = \langle \bar{a}_i, \bar{b}_i^0, \bar{b}_i^1 \rangle : i < \partial \rangle$  satisfies the condition below then for some  $\bar{a} \in {}^\varepsilon M$  we have  $\partial = \sup\{i < \partial : M \models \varphi[\bar{a}, \bar{b}_i^1] \wedge \neg \varphi[\bar{a}, \bar{b}_i^0]\}$
- $\oplus_{M, \varphi, \mathbf{b}}^\partial$  (a)  $\bar{\mathbf{b}} = \langle \bar{a}_i, \bar{b}_i^0, \bar{b}_i^1 \rangle : i < \partial \rangle$ 
  - (b)  $\{\varphi(\bar{x}_\varepsilon, \bar{b}_i^1) \wedge \neg \varphi(\bar{x}_\varepsilon, \bar{b}_i^0) : i < \partial\}$  is included in some  $p \in \mathbf{S}^\varepsilon(M)$
  - (c) for some  $q \in \mathbf{S}^{\varepsilon+\zeta+\zeta}(M)$  for every  $A \in [M]^{<\theta}$ , for every  $i < \theta$  large enough,  $\text{tp}_{\mathbb{L}_{\theta, \theta}(\tau_T)}(\bar{a}_i \wedge \bar{b}_i^0 \wedge \bar{b}_i^1, A, M) = q \upharpoonright M$
  - (d) for every  $\bar{c} \in ({}^{\ell g(\bar{z}_\varepsilon)} M)$  for every  $i < \partial$  large enough,  $\text{tp}_{\mathbb{L}_{\theta, \theta}}(\bar{b}_i^0, \bar{c}, M) = \text{tp}_{\mathbb{L}_{\theta, \theta}}(\bar{b}_i^1, \bar{c}, M)$ .

**Observation 4.6.** If  $M \models T$  then  $\text{inc}_\varphi(M)$  and  $\text{com}_\varphi(M)$  are disjoint.

**Claim 4.7.** If  $\theta \leq \partial = \text{cf}(\partial) \leq \lambda = \lambda^{<\theta}$  and  $|T| \leq \lambda$  then there is  $N$  such that:

- (a)  $N$  is a model of  $T$
- (b)  $N$  is of cardinality  $\lambda$
- (c)  $\partial \in \text{inc}_\varphi(N)$
- (d) if  $\sigma = \text{cf}(\sigma) \in (\theta, \lambda]$  and  $\lambda = \lambda^{[\sigma]}$  (equivalently  $\lambda \geq 2^\sigma \wedge \text{cf}(\lambda) \neq \sigma$ ) then  $\sigma \notin \text{com}_\varphi(N)$ .

*Proof.* Let  $M$  be such that  $(M, \lambda^+, \bar{a}, \mathcal{P})$  are as in 4.2, 4.4 so  $\mathcal{P}$  is  $\lambda^+$ -directed. Clearly  $M$  has cardinality  $> \lambda$ .

Now we choose  $(N_\alpha, u_\alpha)$  by induction on  $\alpha < \partial$  such that:

- (\*) (a)  $N_\alpha \prec_{\mathbb{L}_{\theta, \theta}} M$
- (b)  $N_\alpha$  is of cardinality  $\lambda$
- (c)  $N_\alpha$  includes  $\bigcup_{\beta < \alpha} N_\beta$
- (d)  $u_\alpha \in \mathcal{P}$  includes  $\bigcup_{\beta < \alpha} u_\beta$  and includes  $N_\alpha$
- (e) if  $\alpha = \beta + 1$  then  $\bar{a}_{u_\beta}, \bar{b}_{u_\beta}^0, \bar{b}_{u_\beta}^1$  are  $\subseteq N_\alpha$

- (f) if  $\alpha = \beta + 1, \sigma = \text{cf}(\sigma) \in [\theta, \lambda), \lambda = \lambda^{[\sigma]}$  and  $\oplus_{M, \varphi, \mathbf{b}}^\sigma$  of Definition 4.5 holds then some  $\bar{a} \in {}^\varepsilon(M_\alpha)$  we have  
 $\sigma = \sup\{i < \sigma : M_\alpha \models \varphi[\bar{a}, \bar{b}_i^1] \wedge \neg \varphi[\bar{a}, \bar{b}_i^0]\}.$

There is no problem to carry the induction.  $\square_{4.7}$

**Conclusion 4.8.** Assume  $T$  is not definably stable. If  $\lambda = \lambda^{<\theta} \geq 2^{<\sigma} + |T|$  and  $\sigma > \theta$  then  $\dot{I}(\lambda, T) \geq |\{\partial : \theta \leq \partial < \sigma\}| + 1$  which is always  $\geq 2$ .

*Remark 4.9.* For  $\lambda \in [\theta, 2^\theta)$  we should look more.

*Proof.* By 4.7 and 4.6.  $\square_{4.8}$

**Conjecture 4.10.** 1) If  $\lambda > |T|$  is strong limit and  $\lambda > \theta > \text{cf}(\lambda)$  and  $T$  is not definable stable then  $\dot{I}(\lambda, T) > 1$ .

2) In (1) weaken the assumption on  $\lambda, 2^\theta < \theta < \lambda^\theta$  (hence  $\text{cf}(\theta) < \lambda$ ).

3) Moreover in (2) even  $\geq |\{\partial : \theta \leq \partial, 2^\partial \leq \lambda \text{ and } \partial = \text{cf}(\gamma)\}| + 1$ .

**Discussion 4.11.** 1) In 4.10 necessarily  $\sigma_* := \text{cf}(\lambda) < \theta$  and  $\lambda = \sum_{i < \sigma} \lambda_i, 2^\theta < \lambda_i = \lambda_i^\theta < \lambda$  for some increasing  $\langle \lambda_i : i < \sigma \rangle$ . Let  $\mu_* = \min\{\mu : 2^\mu \geq \lambda\}$ , so  $\mu_* > \theta$  and without loss of generality  $\lambda_i = (\lambda_i)^{<\mu_*}$ . Can we immitate the proof of 4.7 using limit-ultra-power?

2) Let  $N$  be a model of  $T$  of cardinality  $\lambda$  and let  $\bar{N} = \langle N_i : i < \sigma_* \rangle$  be with union  $M$  such that  $\|N_i\| = \lambda_i$  and  $N_i \prec_{\mathbb{L}_{\theta, \theta}(\tau_T)} N$ ; hence  $\bar{N}$  is  $\prec_{\mathbb{L}_{\theta, \theta}(\tau_T)}$ -increasing. Let  $\tau^* = \tau(T) \cup \{P_i : i < \text{cf}(\lambda)\}$ ,  $P_i$  a unary predicate and let  $M_1$  be the  $\tau^+$ -model such that  $M_1^* \upharpoonright \tau_T = M_0 := M$  and  $p_i = |N_i|$  for  $i < \text{cf}(\lambda)$  and let  $M_2$  be an expansion of  $M_1$  with Skolem functions such that  $|\tau(M_2)| = |\theta + \tau(T)|^{<\theta}$  so  $< \lambda$ .

Now if (A) then (B):

- (A) (a)  $D \in \text{uf}_\theta(I)$  and  $M_3 = M_2^I/D, \mathbf{j}$  the can embedding  
 (b)  $A \subseteq M_3$  and  $|A| < \lambda$   
 (c)  $M_4$  is  $\text{Sk}_{M_3}(\mathbf{j}(M_2) \cup A)$   
 (B) (a)  $M_3 = \bigcup_{1 < \text{cf}(\lambda)} \bar{p}_i M_3$   
 (b)  $M_4$  has cardinality  $\lambda$  and is  $\prec_{\mathbb{L}_{\theta, \theta}} M_3$   
 (c)  $\mathbf{j}(M_2) \prec_{\mathbb{L}_{\theta, \theta}} M_3$ .

### § 4(B). More on §3.

**Definition 4.12.** Assume  $\lambda > \theta$  is strong limit of cofinality  $\aleph_0$ .

We say a model  $M$  is  $\lambda$ -special when:

- (a)  $M$  is a model of cardinality  $\lambda$  with  $|\tau(M)| < \lambda$   
 (b)  $\theta \leq \lambda_n < \lambda_{n+1} < \lambda = \sum_k \lambda_k$  and stipulate  $\lambda_{-1} = \theta$   
 (c) we can find  $\langle M_n : n < \omega \rangle$  such that  
 (α)  $M_n \prec_{\mathbb{L}_{\theta, \theta}} M_{n+1}$   
 (β)  $M = \bigcup_n M_n$

- ( $\gamma$ ) letting  $\lambda_n = \|M_n\|$ , for some  $D_n \in \text{uf}_{\lambda_{n-1}, \theta}(\lambda_n)$ ,  $M_n^{\lambda_n}/D_n$  is  $\prec_\theta M_{n+1}$  under the canonical identification.

**Claim 4.13.** 1) Assume  $\lambda > \theta$  is strong limit of cofinality  $\aleph_0$  and  $T$  is a complete theory in  $\mathbb{L}_\theta^1(\tau_T)$ ,  $|T| < \lambda$ . Then  $T$  has exactly one  $\lambda$ -special model (up to isomorphism).

2)  $\mathbb{L}_\theta^1$  has disjoint amalgamation, i.e. if  $M_0 \prec_{\mathbb{L}_\theta^1} M_\ell$  for  $\ell = 1, 2$  that is  $(M_0, c)_{c \in M_0}, (M_\ell, c)_{c \in M}$  has the same  $\mathbb{L}_\theta^1$ -theory and  $|M_1| \cap |M_2| = |M_0|$ , then there is  $M_3$  such that  $M_\ell \prec_{\mathbb{L}_\theta^1} M_3$  for  $\ell = 0, 1, 2$  (hence orbital type are well defined).

Now we can generalize Robinson lemma (hence gives an alternative proof of the interpolation theorem).

**Claim 4.14.** 1) Assume  $\tau_1 \cap \tau_2 = \tau_0$ ,  $T_\ell$  is a complete theory in  $\mathbb{L}_\theta^1(\tau_\ell)$  for  $\ell = 1, 2$  and  $T_0 = T_1 \cap T_2$ . Then  $T_1 \cup T_2$  has a model.

2) We can allow in (1) the vocabularies to have more than one sort.

3) The logic  $\mathbb{L}_\theta^1$  satisfies the interpolation theory.

*Proof.* 1) By 4.13(1).

2) Similarly.

3) Follows as  $\mathbb{L}_\theta^1$  being  $\subseteq \mathbb{L}_{\theta, \theta}$  satisfies  $\theta$ -compactness.  $\square_{4.14}$

*Remark 4.15.* This proof implies the generalization of preservation theorem.

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